Proof Theory of Modal Logic

Lecture 5 : Nested Sequents

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Nested sequents for the S5-cube:
Soundness

Nested sequents in the literature

Independently introduced in:

- [Bull, 1992]; [Kashima, 1994] ⇝ *nested sequents*
- [Brünnler, 2006], [Brünnler, 2009] ⇝ *deep sequents*
- [Poggiolesi, 2008], [Poggiolesi, 2010] ⇝ *tree-hypersequents*

Main references for this lecture:

- [Lellmann & Poggiolesi, 2022 (arXiv)]
- [Brünnler, 2009], [Brünnler, 2010 (arXiv)]
- [Marin & Straßburger, 2014]

Sequent	$\Gamma \Rightarrow \Delta$	Γ, Δ multisets of formulas
One-sided sequent	Γ	Γ multiset of formulas

$$A, B ::= p \mid \overline{p} \mid A \wedge B \mid A \vee B$$

$$\overline{A \wedge B} := \overline{A} \vee \overline{B} \qquad \overline{A \vee B} := \overline{A} \wedge \overline{B}$$

$$A \rightarrow B := \overline{A} \vee B \qquad \bot := p \wedge \overline{p}$$

Rules of G3cp^{one}

$$\text{init}\,\frac{}{\Gamma, p, \overline{p}} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

Exercise. $\vdash_{\mathrm{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathrm{G3cp}^{one}} \overline{\Gamma}, \Delta$, where $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$.

$$A, B ::= p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

$$\overline{A \wedge B} := \overline{A} \vee \overline{B} \qquad \overline{A \vee B} := \overline{A} \wedge \overline{B} \qquad \overline{\Box A} := \Diamond \overline{A} \qquad \overline{\Diamond A} := \Box \overline{A}$$

$$A \to B := \overline{A} \vee B \qquad \bot := p \wedge \overline{p}$$

Nested sequents (denoted $\Gamma, \Delta, \ldots$) are inductively generated as follows:

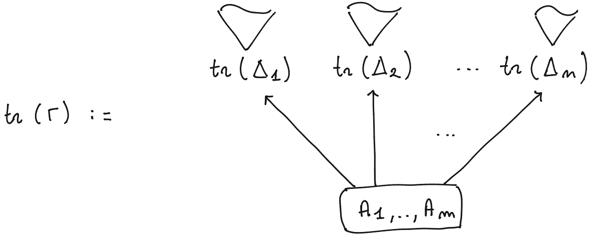
- A multiset of formulas is a nested sequent;
- If Γ and Δ are nested sequents, then Γ, Δ is a nested sequent;
- If Γ is a nested sequent, then $[\Gamma]$ is a nested sequent.
  We call $[\Gamma]$ a boxed sequent.

Nested sequents are multisets of formulas and boxed sequents:

$$\underbrace{A_1, \ldots, A_m}, [\Delta_1], \ldots, [\Delta_n]$$

$$\Gamma = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

To a nested sequent Γ there corresponds the following tree $tr(\Gamma)$, whose nodes $\gamma, \delta, \ldots$ are multisets of formulas:



The formula interpretation $i(\Gamma)$ of a nested sequent Γ is defined as:

- If $m = n = 0$, then $i(\Gamma) := \bot$
- Otherwise, $i(\Gamma) := A_1 \vee \cdots \vee A_m \vee \Box(i(\Delta_1)) \vee \cdots \vee \Box(i(\Delta_n))$

Examples

Examples

1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a)
$$\begin{tikzcd} B_1, B_2 \\ A \arrow[u] \end{tikzcd}$$

b)
$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Examples

1) $\Gamma = A, [B_1, B_2]$

what is $\mathrm{tr}(\Gamma)$?

a)
$$\begin{tikzcd} B_1, B_2 \\ A \arrow[u] \end{tikzcd}$$

✓

b)
$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$A, [B_1], [B_2]$

Examples

1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a)
$$\begin{tikzcd} B_1, B_2 \\ A \arrow[u] \end{tikzcd}$$

b)
$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

2) $\Gamma = A, [B_1, B_2, [C], D]$

what is $tr(\Gamma)$?

a)
$$\begin{tikzcd} C \\ B_1, B_2, D \arrow[u] \\ A \arrow[u] \end{tikzcd}$$

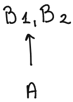
b)
$$\begin{tikzcd} C & & D \\ & B_1, B_2 \arrow[ul] \arrow[ur] & \\ & A \arrow[u] & \end{tikzcd}$$

Examples

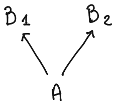
1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a)



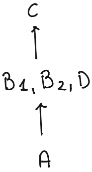
b)



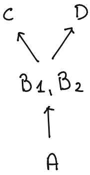
2) $\Gamma = A, [B_1, B_2, [C], D]$

what is $tr(\Gamma)$?

a)



b)



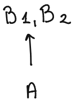
$A, [B_1, B_2, [C], [D]]$

Examples

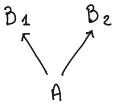
1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a)



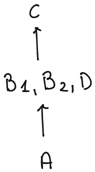
b)



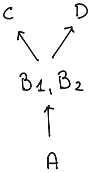
2) $\Gamma = A, [B_1, B_2, [C], D]$

what is $tr(\Gamma)$?

a)



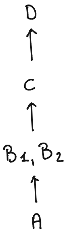
b)



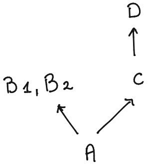
3) $\Gamma = A, [B_1, B_2], [C, [D]]$

what is $tr(\Gamma)$?

a)



b)



Examples

1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a) ✓
$$\begin{tikzcd} B_1, B_2 \\ A \arrow[u] \end{tikzcd}$$

b)
$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

2) $\Gamma = A, [B_1, B_2, [C], D]$

what is $tr(\Gamma)$?

a) ✓
$$\begin{tikzcd} C \\ B_1, B_2, D \arrow[u] \\ A \arrow[u] \end{tikzcd}$$

b)
$$\begin{tikzcd} C & & D \\ & B_1, B_2 \arrow[ul] \arrow[ur] & \\ & A \arrow[u] & \end{tikzcd}$$

3) $\Gamma = A, [B_1, B_2], [C, [D]]$

what is $tr(\Gamma)$?

a)
$$\begin{tikzcd} D \\ C \arrow[u] \\ B_1, B_2 \arrow[u] \\ A \arrow[u] \end{tikzcd}$$

$A[B_1, B_2, [C, [D]]]$

b) ✓
$$\begin{tikzcd} & & \underline{D} \\ \underline{B_1, B_2} & & \underline{C} \arrow[u] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

A context is a nested sequent with one or multiple holes, denoted by $\{\,\}$, each taking the place of a formula in the nested sequent.

- Unary context $\Gamma\{\,\}$
- Binary context $\Gamma\{\,\}\{\,\}$

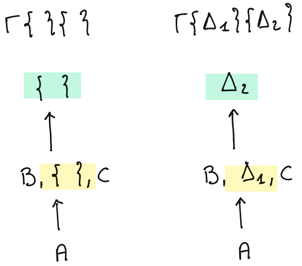
$\Gamma\{\ \}\{\ \} = A, [B, \{\ \}, [\{\ \}], C]$

Δ_1, Δ_2 nested sequents

$\Gamma\{\Delta_1\}\{\Delta_2\} = A, [B, \Delta_1, [\Delta_2], C]$

$\Gamma\{\emptyset\}\{\Delta_2\} = A, [B, [\Delta_2], C]$

$\Gamma\{\Delta_1\}\{\emptyset\} = A, [B, \Delta_1, [\ ], C]$



The depth $depth(\Gamma\{\,\})$ of a unary context $\Gamma\{\,\}$ is defined as:

- $depth(\{\,\}) := 0$;
- $depth(\Gamma\{\,\}, \Delta) := depth(\Gamma\{\,\})$;
- $depth([\Gamma\{\,\}]) := depth(\Gamma\{\,\}) + 1$.

$depth(\Gamma\{\ \}\{\Delta_1\}) = 1$

$depth(\Gamma\{\Delta_1\}\{\ \}) = 2$

$$\text{init}\;\frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\;\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\;\frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\;\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\;\frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Example. Proof of $(\Diamond p \to \Box q) \to \Box(p \to q)$ in NK

$$\vee\;\dfrac{\Box\;\dfrac{\vee\;\dfrac{\wedge\;\dfrac{\Diamond\;\dfrac{\text{init}\;\dfrac{}{\Diamond p, [p, \bar{p}, q]}}{\Diamond p, [\bar{p}, q]} \qquad \Diamond\;\dfrac{\text{init}\;\dfrac{}{\Diamond \bar{q}, [\bar{q}, \bar{p}, q]}}{\Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p} \vee q]}}{\Diamond p \wedge \Diamond \bar{q}, \Box(\bar{p} \vee q)}}{(\Diamond p \wedge \Diamond \bar{q}) \vee \Box(\bar{p} \vee q)}$$

Roadmap

Γ set of formulas, A formula

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$\Gamma \vdash_K A \iff \Gamma \models A$$

$$\vdash \overline{\Gamma}, A$$

NK

NESTED SEQUENTS

Validity of nested sequents [Kuznets & Straßburger, 2018]

For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

A nested sequent Γ is satisfied by an $\mathcal{M}$-map for Γ iff

$$\mathcal{M}, f(\delta) \models B, \text{ for some } \delta \in tr(\Gamma), \text{ for some } B \in \delta$$

A nested sequent Γ is refuted by an $\mathcal{M}$-map for Γ iff

$$\mathcal{M}, f(\delta) \not\models B, \text{ for all } \delta \in tr(\Gamma), \text{ for all } B \in \delta$$

For $X \subseteq \{d, t, b, 4, 5\}$, a nested sequent is X-valid iff it is satisfied by all $\mathcal{M}$-map for Γ, for all models $\mathcal{M}$ satisfying the frame conditions in X.

Lemma. If Γ is derivable in NK then Γ is valid in all Kripke frames.

Proof. By induction on the height of the derivation of Γ. We need to show that initial sequents are valid, and inference rules preserve validity.

Case $\Box$: $$\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\Box$$

To prove: If $\Gamma\{\Box A\}$ is <u>not</u> valid, then $\Gamma\{[A]\}$ is <u>not</u> valid.

Suppose $\Gamma\{\Box A\}$ is not valid, and let $\Box A \in \gamma \in tr(\Gamma\{\Box A\})$.

Then, there is a model $\mathcal{M} = \langle W, R, v \rangle$ and an $\mathcal{M}$-map f for Γ s.t., for all $\eta \in tr(\Gamma)$, for all $B \in \eta$, $\mathcal{M}, f(\eta) \not\models B$.

In particular: $\mathcal{M}, f(\gamma) \not\models \Box A$. Then there is $w \in W$ s.t. $f(\gamma) R w$ and $\mathcal{M}, w \not\models A$.

Let $\delta = \{A\} \in tr(\Gamma\{[A]\})$. Define $g(\delta) = w$, and $g(\eta) = f(\eta)$, for all $\eta \neq \delta$ belonging to $tr(\Gamma\{[A]\})$. It holds that g is an $\mathcal{M}$-map for $\Gamma\{[A]\}$, and it refutes the sequent. $\Box$

$$
\mathsf{d}^\diamond\ \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}}
\qquad
\mathsf{t}^\diamond\ \frac{\Gamma\{\diamond A, A\}}{\Gamma\{\diamond A\}}
\qquad
\mathsf{b}^\diamond\ \frac{\Gamma\{[\Delta, \diamond A], A\}}{\Gamma\{[\Delta, \diamond A]\}}
$$

$$
4^\diamond\ \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\Gamma\{\diamond A, [\Delta]\}}
\qquad
5^\diamond\ \frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0
$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, we write $\mathsf{X}^\diamond$ for the corresponding subset of $\{\mathsf{d}^\diamond, \mathsf{t}^\diamond, \mathsf{b}^\diamond, 4^\diamond, 5^\diamond\}$. We shall consider the calculi $\mathsf{NK} \cup \mathsf{X}^\diamond$.

Example. Proof of $\Box p \to \Box\Box p$ in $\mathsf{NK} \cup \{\cancel{\mathsf{t}}, 4\}$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\diamond\bar{p}, [\diamond\bar{p}, [\diamond\bar{p}, \bar{p}, p]]}\ \mathsf{init}}{\diamond\bar{p}, [\diamond\bar{p}, [\diamond\bar{p}, p]]}\ \mathsf{t}^\diamond}{\diamond\bar{p}, [\diamond\bar{p}, [p]]}\ 4^\diamond}{\diamond\bar{p}, [[p]]}\ 4^\diamond}{\diamond\bar{p}, [\Box p]}\ \Box}{\diamond\bar{p}, \Box\Box p}\ \Box}{\diamond\bar{p} \vee \Box\Box p}\ \vee
$$

Structural rules [Brünnler, 2009]

$$\mathsf{wk}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\ \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Lemma. The rules wk and ctr are hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond}$.

Lemma. All the rules of $\mathsf{NK} \cup \mathsf{X}^{\diamond}$ are hp-invertible.

Proposition. Rule $5^{\diamond}$ is derivable in $\mathsf{NK} \cup \{5^{\diamond}_1, 5^{\diamond}_2, 5^{\diamond}_3\} \cup \{\mathsf{wk}\}$.

$$5^{\diamond}\ \frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

$$5^{\diamond}_1\ \frac{\Gamma\{[\Delta, \diamond A], \diamond A\}}{\Gamma\{[\Delta, \diamond A]\}} \qquad 5^{\diamond}_2\ \frac{\Gamma\{[\Delta, \diamond A], [\Lambda, \diamond A]\}}{\Gamma\{[\Delta, \diamond A], [\Lambda]\}} \qquad 5^{\diamond}_3\ \frac{[\Delta, \diamond A, [\Lambda, \diamond A]]}{\Gamma\{[\Delta, \diamond A, [\Lambda]]\}}$$

Lemma. If Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\diamond}$ then Γ is valid in all X-frames.

Nested sequents with explicit contraction [Marin & Straßburger, 2014]

Rules of $\mathsf{NK}_{\mathsf{ctr}}$

$$\mathsf{init}\ \frac{}{\Gamma\{p,\bar{p}\}} \qquad \mathsf{ctr}\ \frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,}\,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

Rules for extensions

$$\mathsf{d}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,}\,[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,}\,A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{[\Delta,\cancel{\Diamond A}],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$4^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\Diamond A,[\cancel{\Diamond A,}\,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad 5^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A}\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},4,5\}$, we write $\mathsf{X}^{\Diamond}_{\mathsf{ctr}}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond}_{\mathsf{ctr}},\mathsf{t}^{\Diamond}_{\mathsf{ctr}},\mathsf{b}^{\Diamond}_{\mathsf{ctr}},4^{\Diamond}_{\mathsf{ctr}},5^{\Diamond}_{\mathsf{ctr}}\}$.

Lemma. The rule wk is hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Proposition. Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ iff Γ is derivable in $\mathsf{NK}_{\mathsf{ctr}} \cup \mathsf{X}^{\Diamond}_{\mathsf{ctr}}$.

Roadmap

$X \subseteq \{d, t, b, 4, 5\}$

Γ set of formulas, A formula

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$\begin{tikzcd} \Gamma \vdash_{K \cup X} A \arrow[r, Leftrightarrow] & \Gamma \models_X A \arrow[d, dashed, shift right] \\ & \vdash \overline{\Gamma}, A \arrow[u, shift right, "\text{SOUNDNESS}"'] \end{tikzcd}$$

$NK \cup X_\Diamond$

NESTED SEQUENTS

Nested sequents for the S5-cube: Completeness

- Axiom 5, that is, $\Diamond A \to \Box\Diamond A$, is valid in all $\{\mathrm{b}, 4\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$.

Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$

$$
\vee\,\dfrac{\Box\,\dfrac{\Box\,\dfrac{\mathrm{b}^{\Diamond}\,\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}}{\Box\bar{p}, [\Diamond p]}}{\Box\bar{p}, \Box\Diamond p}}{\Box\bar{p} \vee \Box\Diamond p}
$$

$$
\dfrac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}}\ \mathrm{b}^{\Diamond}
$$

$$
\dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ 4^{\Diamond}
$$

- Axiom 5, that is, $\Diamond A \to \Box\Diamond A$, is valid in all $\{\mathrm{b}, 4\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$.
- Axiom 4, that is, $A \to \Box\Box A$, is valid in all $\{\mathrm{t}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{t}^{\Diamond}, 5^{\Diamond}\}$.
- Axiom 4, that is, $A \to \Box\Box A$, is valid in all $\{\mathrm{b}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 5^{\Diamond}\}$.

Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\,\mathrm{b}^{\Diamond}}{\Box\bar{p}, [\Diamond p]}\,\Box}{\Box\bar{p}, \Box\Diamond p}\,\Box}{\Box\bar{p} \vee \Box\Diamond p}\,\vee
$$

For each set of frames characterised by the 5-axioms, there is at least one combination of modal rules which is complete.

For $X \subseteq \{d, t, b, 4, 5\}$, the 45-closure of X is defined as:

$$\hat{X} = \begin{cases} X \cup \{4\} & \text{if } \{b, 5\} \subseteq X \text{ or } \{t, 5\} \subseteq X \\ X \cup \{5\} & \text{if } \{b, 4\} \subseteq X \\ X & \text{otherwise} \end{cases}$$

We say that X is 45-closed if $X = \hat{X}$.

Proposition. For $X \subseteq \{d, t, b, 4, 5\}$ X is 45-closed iff, for $\rho \in \{4, 5\}$, it holds that if ρ is valid in all X-frames, then $\rho \in X$.

To prove:

Theorem (Completeness). For $X \subseteq \{d, t, b, 4, 5\}$, if Γ is X-valid, then Γ is derivable in $\mathsf{NK} \cup \hat{X}^{\Diamond}$.

Solution # 1 - Semantic completeness [Brünnler, 2009]

Theorem (Completeness). For $X \subseteq \{d, t, b, 4, 5\}$, if Γ is X-valid, then Γ is derivable in $NK \cup \hat{X}^{\Diamond}$.

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$\Gamma \vdash_{K \cup X} A \Longleftrightarrow \Gamma \models_X A$$

SOUNDNESS

$$\vdash_{NK \cup \hat{X}^{\Diamond}} \bar{\Gamma}, A$$

NESTED SEQUENTS

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\Diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\Diamond}$.

The proof uses:

- A generalised version of cut (eliminable)

$$\dfrac{\Box\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad \mathsf{tr}^{\Diamond}\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma\{\Box A,[\Delta]\}}{\Gamma\{\Box A,[\Diamond\bar{A},\Delta]\}}\,\mathsf{wk} \qquad \Gamma\{\Diamond\bar{A},[\Diamond\bar{A},\Delta]\}}{\Gamma\{[\Diamond\bar{A},\Delta]\}}\,\mathsf{cut}$$

- Additional structural modal rules (admissible)

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

In the Y-cut:

- $\{\Delta\}^n$ denotes $\overbrace{\{\Delta\}\ldots\{\Delta\}}^{n \text{ times}}$;
- $n \geq 0$;
- $\mathsf{Y} \subseteq \{4, 5\}$;
- there is a derivation of $\Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n$ to $\Gamma\{\Diamond\overline{A}\}\{\emptyset\}^n$ in system $\mathsf{Y}^\Diamond$.

The rank of the cut formula A is defined as the complexity of A, plus one. The cut rank of a derivation is the maximum of the ranks of its cuts.

The notions of cut rank-preserving admissible rule and cut rank-preserving invertible rule are defined analogously to the notions of hp admissible rule and hp invertible rule.

Example: 4-cut

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond \overline{A}\}\{\Diamond \overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

If $\mathsf{Y} = \{4\}$, then $\Gamma\{\ \}\{\ \}^n$ is of the form $\Gamma_1\{\{\ \}, \Gamma_2\{\ \}^n\}$:

$$\text{4-cut}\ \frac{\Gamma_1\{\{\Box A\}, \Gamma_2\{\emptyset\}^n\} \quad \Gamma_1\{\{\Diamond A\}, \Gamma_2\{\Diamond A\}^n\}}{\Gamma_1\{\{\emptyset\}, \Gamma_2\{\emptyset\}^n\}}$$

$$\text{cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad 4^{\Diamond}\ \dfrac{\Gamma\{\Diamond \overline{A}, [\Diamond \overline{A}, \Delta]\}}{\Gamma\{\Diamond \overline{A}, [\Delta]\}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \text{4-cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Gamma\{\Diamond \overline{A}, [\Diamond \overline{A}, \Delta]\}}{\Gamma\{[\Delta]\}}$$

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

If $\mathrm{Y} = \{4\}$, then $\Gamma\{\ \}\{\ \}^n$ is of the form $\Gamma_1\{\{\ \}, \Gamma_2\{\ \}^n\}$:

$$\text{4-cut}\ \frac{\Gamma_1\{\{\Box A\}, \Gamma_2\{\emptyset\}^n\} \quad \Gamma_1\{\{\Diamond A\}, \Gamma_2\{\Diamond A\}^n\}}{\Gamma_1\{\{\emptyset\}, \Gamma_2\{\emptyset\}^n\}}$$

$\Gamma_2\{\ \}^1 := [\{\ \}, \Delta]$

$n = 1$

$\Gamma_1 := \Gamma\{\{\ \}, [\{\ \}, \Delta]\}$

$$\text{cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad 4^{\Diamond}\ \dfrac{\Gamma\{\Diamond\overline{A}, [\Diamond\overline{A}, \Delta]\}}{\Gamma\{\Diamond\overline{A}, [\Delta]\}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \text{4-cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Gamma\{\Diamond\overline{A}, [\Diamond\overline{A}, \Delta]\}}{\Gamma\{[\Delta]\}}$$

Structural modal rules

$$\mathsf{d}^{[\,]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[\,]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{\Delta,[\Sigma]\}}$$

$$4^{[\,]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad 5^{[\,]}\ \frac{\Gamma\{\Delta\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\Delta\}}\ \mathit{depth}(\Gamma\{\,\}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},4,5\}$, we write $\mathsf{X}^{[\,]}$ for the corresponding subset of $\{\mathsf{d}^{[\,]},\mathsf{t}^{[\,]},\mathsf{b}^{[\,]},4^{[\,]},5^{[\,]}\}$.

Example. Proof of $\Diamond A \to \Box\Diamond A$ in $\mathsf{NK} \cup \{\mathsf{b}^{[\,]},4^{[\,]}\}$

$$\vee\ \cfrac{\Box\ \cfrac{\Box\ \cfrac{\mathsf{b}^{[\,]}\ \cfrac{4^{[\,]}\ \cfrac{\Diamond\ \cfrac{\mathsf{init}\ \overline{}}{[[[\bar{p},p],\Diamond p]]}}{[[[\bar{p}],\Diamond p]]}}{[[[\bar{p}]],\Diamond p]}}{[\bar{p}],[\Diamond p]}}{\Box\bar{p},[\Diamond p]}}{\Box\bar{p},\Box\Diamond p}}{\Box\bar{p} \vee \Box\Diamond p}$$

Problem: Rule $\mathsf{d}^{[]}$ is not admissible in the presence of cut.

Solution:

- Show how derivations in $\mathsf{NK} \cup \{\mathsf{t}^{\diamond}, \mathsf{b}^{\diamond}, \mathsf{4}^{\diamond}, \mathsf{5}^{\diamond}\} \cup \{\mathsf{d}^{[]}\} \cup \{\mathsf{cut}\}$ can be transformed into derivations in $\mathsf{NK} \cup \{\mathsf{t}^{\diamond}, \mathsf{b}^{\diamond}, \mathsf{4}^{\diamond}, \mathsf{5}^{\diamond}\} \cup \{\mathsf{d}^{[]}\}$;
- Show that $\mathsf{d}^{[]}$ is admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond}$.

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$:

Lemma (Weakening, Contraction). The rules wk and ctr are height- and cut-rank preserving admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{d}^{[]}\} \cup \{\mathsf{cut}\}$.

Lemma (Invertibility). All the rules of $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{d}^{[]}\} \cup \{\mathsf{cut}\}$ are height- and cut-rank preserving invertible.

Lemma (Admissibility of structural modal rules).

(*i*) Let $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, 4, 5\}$ be 45-closed, and let $\rho \in \mathsf{X}$. Then rule $\rho^{[\,]}$ is cut-rank preserving admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{cut}\}$ and in $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{cut}\} \cup \{\mathsf{d}^{[\,]}\}$.

(*ii*) Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ be 45-closed, and let $\mathsf{d} \in \mathsf{X}$. Then rule $\mathsf{d}^{[\,]}$ is admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond}$.

Proof. Case $\mathsf{b}^{[\,]}$ is admissible in $\mathsf{NK} \cup \{\mathsf{b}^{\diamond}, 4^{\diamond}, 5^{\diamond}\} \cup \{\mathsf{cut}\} \cup \{\mathsf{d}^{[\,]}\}$.

$$\mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{\Delta, [\Sigma]\}}$$

$$\mathsf{b}^{[\,]}\ \frac{4^{\diamond}\ \dfrac{\Gamma\{[\diamond A, \Sigma, [, \diamond A, \Delta]]\}}{\Gamma\{[\diamond A, \Sigma, [\Delta]]\}}}{\Gamma\{\Delta, [\diamond A, \Sigma]\}} \quad \rightsquigarrow \quad 5^{\diamond}\ \frac{\mathsf{b}^{[\,]}\ \dfrac{[\diamond A, \Sigma, [, \diamond A, \Delta]]}{\Gamma\{\Delta, \diamond A, [\diamond A, \Sigma]\}}}{\Gamma\{\Delta, [\diamond A, \Sigma]\}}$$

Reduction Lemma

be 45-closed

Let $X \subseteq \{t, b, 4, 5\}$, and let Y be a subset of $\{4, 5\} \cap X$. Then:

- Let $\mathcal{D}$ be a proof in $\mathsf{NK} \cup X^{\diamond} \cup \{\mathsf{cut}\}$ (or in $\mathsf{NK} \cup X^{\diamond} \cup \{\mathsf{cut}\} \cup \{\mathsf{ser}^{[\,]}\}$) as displayed below, with $cr(\mathcal{D}_1) = cr(\mathcal{D}_2) = p = c(A)$. Then, we can construct the proof $\mathcal{D}^*$ below in the same system, with $cr(\mathcal{D}^*) = p$.

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma\{A\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\overline{A}\}}}{\Gamma\{\emptyset\}}\,\mathsf{cut} \quad \rightsquigarrow \quad \overset{\mathcal{D}^*}{\Gamma\{\emptyset\}}$$

h+1

- Let $\mathcal{D}$ be a proof in $\mathsf{NK} \cup X^{\diamond} \cup \{\mathsf{cut}\}$ (or in $\mathsf{NK} \cup X^{\diamond} \cup \{\mathsf{cut}\} \cup \{\mathsf{ser}^{[\,]}\}$) as displayed below, with $cr(\mathcal{D}_1) = cr(\mathcal{D}_2) = p = c(A)$. Then, we can construct the proof $\mathcal{D}^*$ below in the same system, with $cr(\mathcal{D}^*) = p$.

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma\{\Box A\}\{\emptyset\}^n} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}}{\Gamma\{\emptyset\}\{\emptyset\}^n}\,Y\text{–}\mathsf{cut} \quad \rightsquigarrow \quad \overset{\mathcal{D}^*}{\Gamma\{\emptyset\}}$$

Proof: By induction on the sum of heights of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Two cases of the Reduction Lemma

$$\text{cut}\;\dfrac{\square\;\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\square A,[\Delta]\}} \qquad 4^{\Diamond}\;\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}} \quad\rightsquigarrow\quad \text{4-cut}\;\dfrac{\square\;\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\square A,[\Delta]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{[\Delta]\}}$$

$$\text{4-cut}\;\dfrac{\square\;\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\square A,[[\Sigma]]\}} \qquad \Diamond\;\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}} \quad\rightsquigarrow$$

$$\rightsquigarrow\quad \text{cut}\;\dfrac{4^{[\,]}\;\dfrac{4^{[\,]}\;\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{[[A],[\Sigma]]\}}}{\Gamma\{[[A,\Sigma]]\}} \qquad \text{4-cut}\;\dfrac{\text{wk}\;\dfrac{\square\;\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\square A,[[\Sigma]]\}}}{\Gamma\{\square A,[[\overline{A},\Sigma]]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{[[\overline{A},\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}}$$

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\Diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\Diamond}$.

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$
\begin{tikzcd}
\Gamma \vdash_{K \cup X} A \arrow[rr, Leftrightarrow] \arrow[d, "\text{SYNTACTIC COMPLETENESS}"'] & & \Gamma \models_{X} A \\
\vdash \bar{\Gamma}, A \arrow[rr, "\text{CUT-ELIMINATION}"] & & \vdash \bar{\Gamma}, A \arrow[u, "\text{SOUNDNESS}"']
\end{tikzcd}
$$

$NK \cup X^{\Diamond} \cup \{cut\}$

NESTED SEQUENTS W. CUT

$\underline{\underline{NK \cup \hat{X}^{\Diamond}}}$

NESTED SEQUENTS

Can we get rid of the 45-closure condition?

YES: by adding to NK both the propagation rules $X^{\diamond}$ and the structural rules $X^{[\,]}$. The price to pay is that contraction is no longer admissible.

Theorem. For $X = \{d, t, b, 4, 5\}$, and Γ a set of formulas, it holds that Γ is derivable in $NK_{ctr} \cup X^{\diamond}_{ctr} \cup X^{[\,]}$ iff Γ is X-valid.

Can we get rid of the propagation rules, and use $NK_{ctr} \cup X^{[\,]}$?

NO, some combinations are incomplete, and one example is given in [Marin & Straßburger, 2014].

Conclusions

Comparison

Within the S5-cube ($X \subseteq \{d, t, b, 4, 5\}$):

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
$\mathrm{labK} \cup X$	no	yes	yes	yes, for most	yes, easy!	yes
HS5	yes	yes	yes	yes optimal	yes, easy!	no (N.A.)
$\mathrm{NK} \cup X^{\diamond}$	yes	yes	yes	yes	yes	45-clause
G3K	✓	✗	✓	✓	not directly	no (N.A.)

And beyond the S5-cube?

Comparison

Within the S5-cube ($X \subseteq \{d, t, b, 4, 5\}$):

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
labK $\cup$ X	no	yes	yes	yes, for most	yes, easy!	yes
HS5	yes	yes	yes	yes	yes, easy!	no
NK $\cup$ $X^{\Diamond}$	yes	yes	yes	yes	yes	45-clause

And beyond the S5-cube? confluence : $\forall xyz(xRy \wedge xRz \to \exists \kappa(yR\kappa \wedge zR\kappa))$

$$\frac{yR\kappa, zR\kappa, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \kappa!$$

$$\begin{tikzcd} \Delta \quad y \arrow[r] & \kappa \\ \Gamma \quad x \arrow[u] \arrow[r] & z \quad \Sigma \arrow[u] \end{tikzcd}$$

$$\frac{\Gamma, [\Delta, [\ ]], [\Sigma, [\ ]]}{\Gamma, [\Delta], [\Sigma]}$$

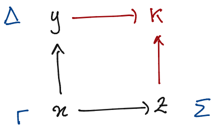
Within the S5-cube ($X \subseteq \{d, t, b, 4, 5\}$):

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
$\mathsf{labK} \cup X$	no	yes	yes	yes, for most	yes, easy!	yes
HS5	yes	yes	yes	yes	yes, easy!	no
$\mathsf{NK} \cup X^{\diamond}$	yes	yes	yes	yes	yes	45-clause

⇒ indexed NS [Marin & Straßburger, 2017]

And beyond the S5-cube? confluence : $\forall xyz(xRy \wedge xRz \to \exists k(yRk \wedge zRk))$

$$\frac{yRk, zRk, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}\; k!$$



$$\frac{\Gamma, [\Delta, [\;]^{i}], [\Sigma, [\;]^{i}]}{\Gamma, [\Delta], [\Sigma]}$$

A few words in conclusion

- There is no good or bad calculus, rather there are different calculi with different properties. The "right" calculus to consider (if there is any) depends on your aim
- Labelled and structured calculi are different but not necessarily opposite or incompatible approaches
  - In some cases, mutual translations between labelled and structured sequents, labelled and structured derivations
  - Possibility to combine labels and structure in the same calculus
- We have presented the most standard (and possibly simplest) extensions of the sequent calculus. However, once established that one can extend the language or the structure, there is no limit to imagination: 2-sequents, display calculus, sequents with histories, linear nested sequents, grafted hypersequents, etc.
- We have presented labelled and structured calculi for the S5 cube of normal modal logics because it is a well-known family of modal logics, and it is the context where this solutions have been initially developed. However, the same or similar solutions have been applied to many other kinds of logics: non-normal modal logics, intuitionistic modal logics, conditional logics, temporal logics, intermediate logics, etc.

Questions?

	fml. interpr.	invertible rules	analyti- city	termination proof search	counterm. constr.	modu- larity
labK $\cup$ X	no	yes	yes	yes, for most	yes, easy!	yes
HS5	yes	yes	yes	yes	yes, easy!	no
NK $\cup$ $X^{\Diamond}$	yes	yes	yes	yes	yes	45-clause

A few words in conclusion

- Questions, suggestions, discussion etc. are very welcome

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- Thank you for attending, we hope you enjoyed the course