Proof Theory of Modal Logic

Lecture 4: Hypersequent calculi

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- Lecture 1: Sequent calculi
- Lecture 2: Labelled sequent calculi
- In this lecture we start looking at structured calculi, that extend sequent calculi with additional structural connectives

In particular, we now look at hypersequent calculi

- Simple generalisation of sequent calculi
- Introduced by [Mints, 1968] [Pottinger, 1983], [Avron, 1987] to provide cut-free calculi for modal and relevant logics

☞ In this lecture we focus only on modal logic S5

Axiomatisation of S5

$$\mathsf{K} + \begin{array}{ll} \mathsf{t} & \Box A \to A \\ \mathsf{4} & \Box A \to \Box\Box A \\ \mathsf{b} & A \vee \Box\neg\Box A \end{array} \quad \text{or} \quad \mathsf{K} + \begin{array}{ll} \mathsf{t} & \Box A \to A \\ \mathsf{5} & \Box A \vee \Box\neg\Box A \end{array}$$

Semantics of S5

Kripke models with equivalence relation

Complexity of S5

The validity/derivability problem for S5 is coNP-complete

Recap

- No cut-free, Gentzen-style sequent calculus for S5 (Lecture 1)
- Cut-free labelled calculus for S5 (Lecture 2)
- What about an internal, structured calculus for S5?

A hypersequent calculus for S5

Main reference for this calculus

- *A cut-free simple sequent calculus for modal logic S5* [Poggiolesi, 2008]: Definition of the calculus and structural analysis

Further references

- [Lellmann, 2016]: Optimal proof-search procedure in the calculus
- [Restall, 2007]: A version of the calculus with explicit structural rules

Hypersequent Finite multiset of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ are the components of the hypersequent

Formula interpretation

$$i(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$$
$$=$$
$$\Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Delta_n)$$

☞ Differently from labelled sequents, hypersequents can be interpreted as formulas

Initial hypersequents and propositional hypersequent rules

$$\text{init} \quad p, \Gamma \Rightarrow \Delta, p \qquad \rightsquigarrow \qquad \text{init} \quad \mathcal{H} \mid p, \Gamma \Rightarrow \Delta, p$$

$$\vee_{\mathsf{R}} \ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \rightsquigarrow \qquad \vee_{\mathsf{R}} \ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

Modal rules for S5

$$\Box_{\mathsf{L}} \ \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \Box_{\mathsf{L}}^{\mathsf{t}} \ \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta}$$

$$\Box_{\mathsf{R}} \ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

Example. Derivation of axiom B

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \mid \Box A \Rightarrow}{\Rightarrow A \mid \Box A \Rightarrow}\ \Box_L}{\Rightarrow A \mid \Rightarrow \neg\Box A}\ \neg_R}{\Rightarrow A, \Box\neg\Box A}\ \Box_R}{\Rightarrow A \vee \Box\neg\Box A}\ \vee_R$$

Exercise. Derive axioms k, t, 4, 5

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\vdash_{S5} i(\mathcal{H})$

Proof sketch (i). We consider simple instances of the rules

$$\Box_L \frac{\Box A \Rightarrow | A \Rightarrow B}{\Box A \Rightarrow | \Rightarrow B}$$

i.	$\vdash \Box\neg\Box A \vee \Box(A \to B)$	$(i(P))$
ii.	$\vdash \Box\neg\Box A \vee \neg\Box\neg\Box A$	(CPL)
iii.	$\vdash \neg\Box\neg\Box A \to \Box A$	(axiom 5)
iv.	$\vdash \Box A \wedge \Box(A \to B) \to \Box B$	(axiom k)
v.	$\vdash \Box\neg\Box A \vee \Box B = i(C)$	(by classical reasoning)

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\vdash_{\mathrm{S5}} i(\mathcal{H})$

Proof sketch (ii). We consider simple instances of the rules

$$\Box_R \frac{B \Rightarrow C \mid \Rightarrow A}{B \Rightarrow C, \Box A}$$

i.	$\vdash \Box(B \to C) \lor \Box A$	($i(P)$)
ii.	$\vdash (B \to C) \to (B \to C \lor \Box A)$	(CPL)
iii.	$\vdash \Box(B \to C) \to \Box(B \to C \lor \Box A)$	(ii, by K valid rule)
iv.	$\vdash \Box A \to (B \to C \lor \Box A)$	(CPL)
v.	$\vdash \Box\Box A \to \Box(B \to C \lor \Box A)$	(iv, by K valid rule)
vi.	$\vdash \Box A \to \Box\Box A$	(axiom 4)
vii.	$\vdash \Box A \to \Box(B \to C \lor \Box A)$	(from iv, vi)
viii.	$\vdash \Box(B \to C \lor \Box A) = i(C)$	(from i, iii, vii)

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vi.	$\vdash \Box A \to \Box\Box A$	(axiom 4)
vii.	$\vdash \Box A \to \Box(B \to C \vee \Box A)$	(from iv, vi)
viii.	$\vdash \Box(B \to C \vee \Box A) = i(C)$	(from i, iii, vii)

Exercise. Prove soundness of all the rules of **HS5**

In order to syntactically prove the completeness of **HS5**, we need to analyse its structural properties

Relevant structural properties:

1. Hp-invertibility of all rules
2. Hp-admissibility of weakening and contraction
3. Admissibility of cut

- ☞ Interesting properties on their own
- ☞ Some dependeces
  - ▶ Hp-admissibility of contraction depends on 1.
  - ▶ Admissibility of cut depends on 1. and 2.

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(base case) If $h = 0$, then the conclusion $\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A$ is an initial hypersequent. There are three possibilities:

1. $\mathcal{H}$ is an intial hypersequent
2. $p \in \Gamma \cap \Delta$ for some p
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In each of these cases, the premiss $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ is an initial hypersequent, hence it is derivable with height 0.

(inductive step) If $h > 0$, we need to consider the last rule application in the derivation of the conclusion $\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A$.

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1. $\Box A$ is principal in the last rule application
2. $\Box A$ is not principal in the last rule application

(case 1.) If $\Box A$ is principal in the last rule application, then the last rule application is precisely

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A} \Box_R$$

which means that the premiss $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ has a derivation of height $h - 1$.

(case 2.) If $\Box A$ is not principal in the last rule application, then, since $\mathcal{H}, \Gamma, \Delta$ can be any hypersequent and multisets, the last rule applied can be any rule of the calculus, hence one needs to consider all of them...

(case 2.) If $\Box A$ is not principal in the last rule application, then, since $\mathcal{H}, \Gamma, \Delta$ can be any hypersequent and multisets, the last rule applied can be any rule of the calculus, hence one needs to consider all of them... Here we consider as an example the case where the last rule applied is $\Box_R$ itself.

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$$\frac{\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B}{\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

where $(\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B) = \mathcal{H}$

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(Alternatively, one can have $\Box B \in \Delta$, the proof is analogous in this case.)

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(Alternatively, one can have $\Box B \in \Delta$, the proof is analogous in this case.)

Then, by the hp-invertibility of $\Box_R$, that holds at height $h - 1$ by inductive hypothesis, $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Rightarrow A$ has a derivation $\mathcal{D}$ of height $h' \leq h - 1$.

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$$\frac{\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B}{\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta, \Box A} \Box_R$$

where $(\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B) = \mathcal{H}$ and the premiss $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B$ has a derivation of height $h - 1$.

(Alternatively, one can have $\Box B \in \Delta$, the proof is analogous in this case.)

Then, by the hp-invertibility of $\Box_R$, that holds at height $h - 1$ by inductive hypothesis, $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Rightarrow A$ has a derivation $\mathcal{D}$ of height $h' \leq h - 1$. Therefore, by extending $\mathcal{D}$ with an application of $\Box_R$ to this hypersequent, we obtain a derivation of height $h' + 1 \leq h$ of $\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$, that is, $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$.

qed

Structural rules

$$\mathsf{wk_L}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta} \qquad \mathsf{wk_R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathsf{ctr_L}\ \frac{\mathcal{H} \mid A, A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr_R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathsf{wk_{ext}}\ \frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr_{ext}}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}$$

☞ Note: external forms of weakening and contraction

The cut rule

$$\mathsf{cut}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{H}' \mid A, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Theorem. Left, right and external weakening and contraction are hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

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Hint. In order to prove the hp-admissibility of some structural rules you may need the following (nice) rule

$$\text{merge}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

Theorem. The rule merge is hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

Theorem. Cut is admissible in **HS5**

Proof sketch. By induction on the complexity of the cut formula and subinduction on the cut height.

As an example, consider the following derivation, with the cut formula $\Box A$ principal in the last rule application in both premisses of cut

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}\,\Box_R \qquad \dfrac{\mathcal{H}' \mid A, \Box A, \Gamma' \Rightarrow \Delta'}{\mathcal{H}' \mid \Box A, \Gamma' \Rightarrow \Delta'}\,\Box^t_L}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}$$

Converted into the following, with one application of cut at a lower height, and one application of cut with a cut formula of lower complexity

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \qquad \dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \qquad \mathcal{H}' \mid A, \Box A, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma', A \Rightarrow \Delta, \Delta'}\,\text{cut}}{\mathcal{H} \mid \mathcal{H} \mid \mathcal{H}' \mid \Gamma \Rightarrow \Delta \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}}{\mathcal{H} \mid \mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{wk}^*}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{ctr}_{\text{ext}} \times 2$$

(where wk^* denotes multiple applications of (left and right) weakening

Completeness

Theorem. If $\vdash_{S5} A$, then $\vdash_{\mathbf{HS5}} \Rightarrow A$

Proof sketch.

- All axioms of S5 are derivable in **HS5** (*exercise*)
- The necessitation rule is admissible in **HS5** (*exercise*)
- Modus ponens is simulated by cut

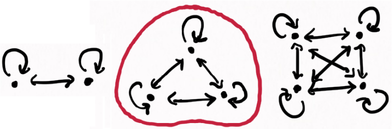
So far, purely syntactical analysis.
What about a semantics for the calculus?

Two semantics for S5

1. Kripke models with equivalence relation, or
2. Universal semantics $\mathcal{M} = \langle W, v \rangle$
   - No binary relation
   - $\mathcal{M}, w \Vdash \Box A$ *iff* for all $u \in W$, $\mathcal{M}, u \Vdash A$

☞ $\Box A$ true somewere iff A true everywhere

☞ Corresponds to choosing one cluster of a model with equivalence relation



Notation. We denote $\mathcal{U}$ the class of all universal models

- Different components $\rightsquigarrow$ different worlds
- For each component, formulas on the left true, formulas on the right false in the corresponding world

$$\Box_L \ \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \uparrow$$

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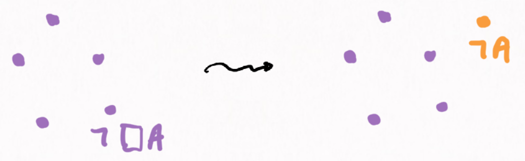

Valid hypersequent $\mathcal{M} \models \mathcal{H}$ *iff* $\exists\, \Gamma \Rightarrow \Delta \in \mathcal{H} : \mathcal{M} \models \Gamma \Rightarrow \Delta$

- ☞ Semantically, a hypersequent is a disjunction of validities
- ☞ That is, $\mathcal{H}$ valid iff one component is valid

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\models_{\mathcal{U}} \mathcal{H}$

Proof sketch. One needs to show that the inital hypersequents are valid in $\mathcal{U}$ (trivial) and that all rules of **HS5** preserve validity in universal models (*exercise*).

We now prove the opposite direction (completeness of **HS5**), namely that

$$\text{if } \models_{\mathcal{U}} \mathcal{H}, \text{ then } \vdash_{\mathbf{HS5}} \mathcal{H}$$

1. First, we define a terminating (optimal) proof-search procedure in **HS5**
2. Then, we show that every failed proof constructed according to this procedure provides a countermodel of the root hypersequent: that is, if $\nvdash_{\mathbf{HS5}} \mathcal{H}$, then $\not\models_{\mathcal{U}} \mathcal{H}$

A proof-search procedure in **HS5**

Main reference [Lellmann, 2016]

As a first step, we consider a cumulative formulation of **HS5**

Comulative formulation of a rule

The principal formula is copied to the premiss(es)

e.g.

$$\vee_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

$$\vee_L \frac{\mathcal{H} \mid A, A \vee B, \Gamma \Rightarrow \Delta \qquad \mathcal{H} \mid B, A \vee B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \vee B, \Gamma \Rightarrow \Delta}$$

$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

☞ Remark. The rules $\Box_L$ and $\Box_L^t$ are already in cumulative form

Notation. We call $\mathbf{HS5_{cum}}$ the calculus defined by the cumulative formulation of the rules of **HS5**

Theorem (Soundness). If $\vdash_{\mathbf{HS5_{cum}}} \mathcal{H}$, then $\models_{\mathcal{U}} \mathcal{H}$

Proof. The cumulative rules are admissible in **HS5**.

Example: Admissibility of the cumulative version of $\Box_R$ in **HS5**

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \mid \Rightarrow A}\ \text{by invertibility of } \Box_R}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}\ \text{ctr}_{\text{ext}}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

Therefore: if $\vdash_{\mathbf{HS5_{cum}}} \mathcal{H}$, then $\vdash_{\mathbf{HS5}} \mathcal{H}$, hence $\models_{\mathcal{U}} \mathcal{H}$

Clearly, the complexity of hypersequents is not reduced by backward applications of cumulative rules

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Local loop-checking condition (LLCC)

An application of a hypersequent rule with premiss $\mathcal{G}$, or premisses $\mathcal{G}_1$ and $\mathcal{G}_2$, and conclusion $\mathcal{H}$ satisfies the local loop checking condition if for each premiss $\mathcal{G}_i$, there exists a component $\Gamma \Rightarrow \Delta$ in $\mathcal{G}_i$ such that for no component $\Sigma \Rightarrow \Pi$ of the conclusion $\mathcal{H}$ we have $set(\Gamma) \subseteq set(\Sigma)$ and $set(\Delta) \subseteq set(\Pi)$

Example: the following rule applications violate the LLCC

$$\dfrac{\Rightarrow p \wedge q, q, p \qquad \Rightarrow p \wedge q, q, q}{\Rightarrow p \wedge q, q}\ \wedge_R \qquad \dfrac{p \Rightarrow q \mid r \Rightarrow \Box q \mid \Rightarrow q}{p \Rightarrow q \mid r \Rightarrow \Box q}\ \Box_R$$

☞ The LLCC prevents the applications of rules that do not add additional information to the hypersequents

Saturated hypersequent

A hypersequent which is not initial and such that no rule is backward applicable to it without violating the LLCC

Backward proof-search with LLCC for $\mathcal{H}$

The construction of a derivation tree from the root to the leaves such that the root is labelled with the hypersequent $\mathcal{H}$, and the branches are expanded by applying at each step a backwards applicable rule that satisfies the LLCC. The construction terminates when all leaves are labelled with hypersequents that are either initial or saturated

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☞ In particular, we show that from every failed proof for $\mathcal{H}$ we can extract a countermodel of $\mathcal{H}$

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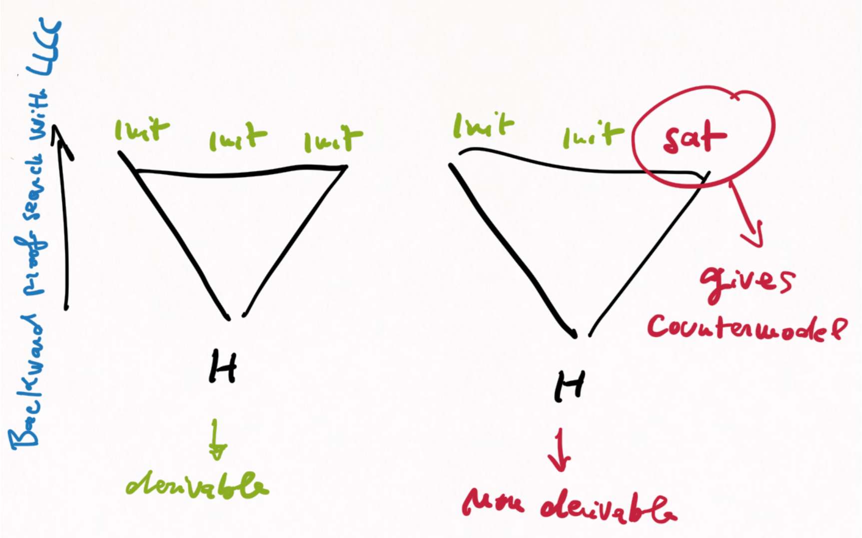
☞ Is proof-search with LLCC still complete?

☞ We now prove that proof-search with LLCC is complete by showing that every hypersequent $\mathcal{H}$ on which it fails is not valid in the universal semantics

☞ In particular, we show that from every failed proof for $\mathcal{H}$ we can extract a countermodel of $\mathcal{H}$

☞ More precisely, we show that each saturated hypersequent occurring in a failed proof of $\mathcal{H}$ provides the information needed to build such countermodel

The procedure in a picture



Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$

Countermodel extracted from a saturated hypersequent

We define $\mathcal{M} = \langle W, v \rangle$ on the basis of $\mathcal{H}$ as follows

- $W = \{k \mid \Gamma_k \Rightarrow \Delta_k \in \mathcal{H}\}$
- For all $p \in Atm$, $v(p) = \{k \in W \mid p \in \Gamma_k\}$

Countermodel lemma

For all formulas A, for all components $\Gamma_k \Rightarrow \Delta_k$,

- if $A \in \Gamma_k$, then $k \Vdash A$
- if $A \in \Delta_k$, then $k \nVdash A$

Proved by induction on the construction of A (*exercise*)

Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$, and $\mathcal{M}$ be the model defined on the basis of $\mathcal{H}$ as in the previous slide

The countermodel lemma implies that

- for all $\Gamma_k \Rightarrow \Delta_k \in \mathcal{H},\ k \nVdash \bigwedge \Gamma_k \to \bigvee \Delta_k$
- hence, $\mathcal{M} \not\models \mathcal{H}$

Moreover, since all rules are cumulative, we have

for all $\Sigma \Rightarrow \Pi \in \mathcal{G},$ there is $\Gamma_k \Rightarrow \Delta_k \in \mathcal{H}$ s.t. $\Sigma \subseteq \Gamma_k$ and $\Pi \subseteq \Delta_k$

Therefore

- for all $\Sigma \Rightarrow \Pi \in \mathcal{G},$ there is $k \in W$ s.t. $k \nVdash \bigwedge \Sigma \to \bigvee \Pi$
- hence, $\mathcal{M} \not\models \mathcal{G}$

☞ $\mathcal{M}$ is a countermodel of the root hypersequent $\mathcal{G}$

Example of failed proof

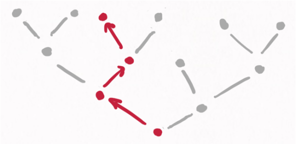
$$w_1 \qquad\qquad w_2 \qquad\qquad w_3$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, p \lor q, \ldots \Rightarrow \ldots \mid q, \ldots \Rightarrow p \mid p, \ldots \Rightarrow q \quad \times}{\checkmark \quad p \lor q, \ldots \Rightarrow \ldots \mid q, p \lor q \Rightarrow p \mid p, p \lor q \Rightarrow q}\lor_L}{\checkmark \quad p \lor q, \Box(p \lor q) \Rightarrow \ldots \mid q, p \lor q \Rightarrow p \mid p \lor q \Rightarrow q}\lor_L}{p \lor q, \Box(p \lor q) \Rightarrow \Box p \lor \Box q, \Box p, \Box q \mid p \lor q \Rightarrow p \mid p \lor q \Rightarrow q}\lor_L}{p \lor q, \Box(p \lor q) \Rightarrow \Box p \lor \Box q, \Box p, \Box q \mid p \lor q \Rightarrow p \mid \Rightarrow q}\Box_L}{p \lor q, \Box(p \lor q) \Rightarrow \Box p \lor \Box q, \Box p, \Box q \mid \Rightarrow p \mid \Rightarrow q}\Box_L}{\Box(p \lor q) \Rightarrow \Box p \lor \Box q, \Box p, \Box q \mid \Rightarrow p \mid \Rightarrow q}\Box_L^t}{\Box(p \lor q) \Rightarrow \Box p \lor \Box q, \Box p, \Box q \mid \Rightarrow p}\Box_R}{\Box(p \lor q) \Rightarrow \Box p \lor \Box q, \Box p, \Box q}\Box_R}{\Box(p \lor q) \Rightarrow \Box p \lor \Box q}\lor_R}{}$$

$$w_1, w_2, w_3 \Vdash \Box(p \lor q)$$
$$w_1, w_2, w_3 \not\Vdash \Box p \lor \Box q$$

Theorem. Backward proof-search with LLCC in **HS5** provides a NP decision procedure for non derivability in S5

☞ At each step, non deterministically chose an applicable rule satisfying the LLCC and a correct premiss



This result relies on two key remarks:

1. The length of branches in a proof built by backward proof-search with LLCC is polynomially bounded by the length of the root hypersequent (see next slide)
2. Verifying the LLCC takes polynomial time

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof.

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Sketch of proof. The length of a branch depends on the number of rule applications in that branch.

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula.

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Therefore, the length of hypersequents in the proof, hence the length of branches, is in $\mathcal{O}(n^2)$ qed

Mono- vs. Multi-modal logics

How many modalities can sequent calculi support?

Sequent and labelled sequent calculi can be extended to multimodal logics without essential modifications.

Example. Let K_n be the logic with n K-modalities $\Box_1, \dots, \Box_n$. The calculus $\mathbf{G3K_n}$ can be defined considering, for each $i \leq n$, the rule

$$\mathrm{k_i}\ \frac{\Gamma \Rightarrow A}{\Gamma', \Box_i\Gamma \Rightarrow \Box_i A, \Delta}$$

Similary, a labelled calculus for K_n can be defined considering relational symbols $R_1, \dots, R_n$ and, for each $i \leq n$, the rules

$$\Box_\mathrm{L}\ \frac{xR_iy, x : \Box_i A, y : A, \Gamma \Rightarrow \Delta}{xR_iy, x : \Box_i A, \Gamma \Rightarrow \Delta} \qquad \Box_\mathrm{R}\ \frac{xR_iy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_i A}\ (y!)$$

☞ The properties of the sequent and the labelled calculus for K hold also for the sequent and the labelled calculus for K_n

The same is not possible in **HS5**

- The hypersequent construct | can represent only one S5 modality
- After all, a model can have only one universal modality

However, the universal modality can be combined with other kinds of modalities

Example.
Let $\mathsf{K}_{\mathcal{U}}$ be the logic with a K modality $\Box$ and a universal modality $\blacksquare$

Semantics $\mathcal{M} = \langle W, R, v \rangle$, with

- $\mathcal{M}, w \Vdash \Box A$ iff for all u s.t. wRu, $\mathcal{M}, u \Vdash A$
- $\mathcal{M}, w \Vdash \blacksquare A$ iff for all u, $\mathcal{M}, u \Vdash A$

(redundant but complete) (cf. [Goranko, Passy, 1992] for a more detailed analysis)

Axiomatisation

- K axiomatisation for $\Box$
- S5 axiomatisation for $\blacksquare$
- $\blacksquare A \to \Box A$

Hypersequent calculus S5 hypersequent calculus for $\blacksquare$, extended with the hypersequent formulation of the rule k for $\Box$:

$$\mathsf{k}\ \frac{\mathcal{H} \mid \Sigma \Rightarrow A}{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Exercise. Derive the axiom $\blacksquare A \to \Box A$

As we have seen, a sequent $\Gamma \Rightarrow \Delta$ represents a consequence relation between the antecedent Γ (the assumptions) and the consequent Δ

But... which kind of assumptions?

Global vs. local modal consequence relation

Syntactically $\Gamma \vdash A$ (Hilbert systems)

- Global Both propositional and modal rules (necessitation) can be applied to the assumptions
- Local Only propositional rules be applied to the assumptions

Semantically $\Gamma \models A$

- Global For all $\mathcal{M}$, $\mathcal{M} \models \bigwedge \Gamma$ implies $\mathcal{M} \models A$
- Local For all $\mathcal{M}$, for all w, $\mathcal{M}, w \Vdash \bigwedge \Gamma$ implies $\mathcal{M}, w \Vdash A$

Remark.

- The sequent rule

$$\to_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

  expresses the deduction theorem, that holds (in this form) for local consequence only

$$A \models_{local} B \;\rightsquigarrow\; \models_{local} A \to B$$
$$A \models_{global} B \;\not\rightsquigarrow\; \models_{global} A \to B$$

  e.g. $A \models_{global} \Box A$ but $\not\models_{global} A \to \Box A$

- Indeed, validity of modal sequents is defined exactly as the local consequence

☞ Modal sequents represent local consequence relations

The hypersequent calculus can be used to reasoning under global assumptions

Indeed, reasoning under global assumptions in K:

$$B_1, ..., B_n \vdash_{global} A$$

can be reduced to

$$\vdash_{K_{\mathcal{U}}} \blacksquare B_1 \wedge ... \wedge \blacksquare B_n \rightarrow A$$

which is expressed in $\mathbf{HK_U}$ with the sequent

$$\blacksquare B_1, \ldots, \blacksquare B_n \Rightarrow A$$

☞ We now show that $\mathbf{HK_U}$ provides a decision procedure for reasoning under global assumptions in K

Decision procedure analogous to **HS5**:

- Cumulative formulation of all rules of the calculus

  example: $$\text{k}\ \frac{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta \mid \Sigma \Rightarrow A}{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

- Loop checking and proof-search strategy defined as for **HS5**
- Termination of proof search: by measuring the size of maximal hypersequents in proof-search. Remark: exponential size!
- Completeness of proof-search: countermodel from every saturated hypersequent (next slide)

Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$ in $\mathbf{HK_U}$

Countermodel extracted from a saturated hypersequent

We define $\mathcal{M} = \langle W, R, v \rangle$ on the basis of $\mathcal{H}$ as follows

- $W = \{k \mid \Gamma_k \Rightarrow \Delta_k \in \mathcal{H}\}$
- For all $k, \ell \in W$, $kR\ell$ iff $\Box A \in \Gamma_k$ or $\blacksquare A \in \Gamma_k$, then $A \in \Gamma_\ell$
- For all $p \in Atm$, $v(p) = \{k \in W \mid p \in \Gamma_k\}$

Countermodel lemma

For all formulas A, for all components $\Gamma_k \Rightarrow \Delta_k$,

- if $A \in \Gamma_k$, then $k \Vdash A$
- if $A \in \Delta_k$, then $k \nVdash A$

Proved by induction on the construction of A (*exercise*)

Exercise. Prove that $\Box q$ is derivable under assumptions p and $p \to q$ if and only if both assumptions are global

Exercise. Prove that $\Box q$ is derivable under assumptions p and $p \to q$ if and only if both assumptions are global

Possible solution

- $\Box q$ is derivable under global assumptions p and $p \to q$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\checkmark}{\ldots \mid p, p \to q \Rightarrow q, p} \qquad \overset{\checkmark}{\ldots \mid q, p, p \to q \Rightarrow q}}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q \mid p, p \to q \Rightarrow q} \to_L}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q \mid p \Rightarrow q} \blacksquare_L}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q \mid \ \Rightarrow q} \blacksquare_L}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q} \Box_R$$

Possible solution

- $\Box q$ is not derivable under global assumption p and local assumption $p \to q$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\checkmark}{p, \blacksquare p, p \to q, p \Rightarrow \Box q \mid p \Rightarrow q} \qquad \overset{\times}{q, p, \blacksquare p, p \to q \Rightarrow \Box q \mid p \Rightarrow q}}{p, \blacksquare p, p \to q \Rightarrow \Box q \mid p \Rightarrow q}\ {\to_R}}{\blacksquare p, p \to q \Rightarrow \Box q \mid p \Rightarrow q}\ \blacksquare^t_L}{\blacksquare p, p \to q \Rightarrow \Box q \mid \ \Rightarrow q}\ \blacksquare_L}{\blacksquare p, p \to q \Rightarrow \Box q}\ \Box_R
$$

$$
\overset{\mathbf{1}}{q, p, \blacksquare p, p \to q \Rightarrow \Box q} \mid \overset{\mathbf{2}}{p \Rightarrow q}
$$

$$
W = \{1, 2\} \qquad 1R1, 1R2 \qquad v(p) = \{1, 2\}, v(q) = \{1\}
$$

☞ $1 \Vdash \blacksquare p,\ 1 \Vdash p \to q,\ 1 \nVdash \Box q$

Several alternative hypersequent calculi for S5: [Mints, 1971], [Pottinger, 1983], [Avron, 1993], [Restall, 2007], [Poggiolesi, 2008], [Kurokawa, 2013], [Lahav, 2013]

A nice and influential calculus: [Avron, 1993]

$$\mathsf{T}\ \frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta} \qquad 4\ \frac{\mathcal{H} \mid \Box\Gamma \Rightarrow A}{\mathcal{H} \mid \Box\Gamma \Rightarrow \Box A}$$

$$\mathsf{MS}\ \frac{\mathcal{H} \mid \Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi}{\mathcal{H} \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Sigma \Rightarrow \Pi}$$

- “Modular” extension of a sequent calculus for S4
- S5 obtained with the addition of a hypersequential structural rule: the Modal Splitting (MS)

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Appendix:
Labelled vs. structured calculi.
Two separate worlds?

Labelled calculus (informal definition)
Any calculus which includes linguistic components that do not belong to the language of the logic

A simple labelled calculus: **LabS5**

- Labels $x, y, z, \ldots$
- Labelled formulas $x : A$
- No relational atoms
- Rules of **G3cp** enriched with labels $\vee_R \dfrac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}$
- Modal rules

$$\Box_L \frac{y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} (y \in \Gamma, \Delta) \qquad \Box_R \frac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A} (y!)$$

Remark. **LabS5** notational variant of predicate calculus

$$\Box_L \frac{y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} (y \in \Gamma, \Delta) \qquad \Box_R \frac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A} (y!)$$

$$\rightsquigarrow \qquad\qquad \rightsquigarrow$$

$$\forall_L \frac{A(y/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} (y \in \Gamma, \Delta) \qquad \forall_R \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A} (y!)$$

☞ S5 corresponds to the "uniform monadic first-order predicate calculus" [Prior, Fine, 1977]

- Relational symbols with only one argument
- Formulas with at most one free variable

*Prior, Fine, Worlds, Times and Selves, Univ. Mass. Press, 1977.

- Mutual translations between semantically equivalent hyper- and labelled sequents

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$
$$\updownarrow$$
$$x_1 : \Gamma_1, \ldots, x_n : \Gamma_n \Rightarrow x_1 : \Delta_1, \ldots, x_n : \Delta_n$$

- Direct correspondence between hypersequent and labelled sequent rule applications

$$\begin{array}{ccc} \text{hyp}(\Box_\text{L}, \Box^\text{t}_\text{L}) & \quad & \text{hyp}(\Box_\text{R}) \\ (x \neq y) \updownarrow (x = y) & & \updownarrow \\ \text{lab}(\Box_\text{L}) & & \text{lab}(\Box_\text{R}) \end{array}$$

- Direct correspondence between hypersequent and labelled sequent derivations