Proof Theory of Modal Logic

Lecture 1: Sequent calculi

Tiziano Dalmonte & Marianna Girlando

Free University of Bozen-Bolzano
ILLC, Universtiy of Amsterdam

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Proof theory   The study of logics from the point of view of their proofs (or their proof systems)

- practice: towards automated reasoning methods
- theory: from properties of proofs to properties of logics

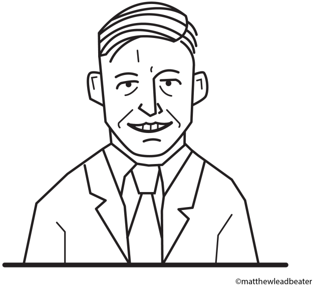
Many types of proof systems: Hilbert systems, natural deduction, tableaux, resolution, ...

In this course we focus on sequent-style calculi for modal logics

- Lecture 1: Sequent calculi
- Lecture 2: Labelled calculi
- Lecture 3: Hypersequent calculi
- Lecture 4: Nested sequent calculi
- Lecture 5: Beyond the modal cube

- Quick intro to sequent calculus
- Our point of view: desiderata on proof systems
- Quick intro to modal logic
- Sequent calculi for modal logics

A quick intro to sequent calculus



Introduced by Gerhard Gentzen in [Gentzen, 1935a]

- As an auxiliary tool for natural deduction normalization
- Provides first proof of decidability of IPL
- End goal [Gentzen, 1935b]: prove the consistency of arithmetic

In sequent calculus, the proof objects are not formulas (like in axiomatic systems or natural deduction) but consequence relations (*the sequents*)

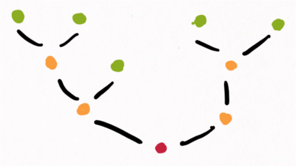
$$\Gamma \Rightarrow \Delta$$

expressing that at least one formula in Δ follows from the assumptions in Γ

Sequent $\quad \Gamma \Rightarrow \Delta \qquad \Gamma, \Delta$ finite sequences of formulas

Derivation of a sequent $\Gamma \Rightarrow \Delta$

Finite tree with each node labelled with a sequent, where the root is labelled with $\Gamma \Rightarrow \Delta$, the leaves are labelled with initial sequents of the form $A \Rightarrow A$, and each internal node is obtained from its children by the application of a sequent rule (see next slide)



Logical rules

$$\neg_L\,\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad \neg_L\,\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \qquad \wedge_L^1\,\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_L^2\,\frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

$$\wedge_R\,\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \qquad \vee_L\,\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^1\,\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\vee_R^2\,\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \rightarrow_L\,\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R\,\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Structural rules

$$\mathrm{wk}_L\,\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathrm{wk}_L\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \mathrm{ctr}_L\,\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$$

$$\mathrm{ctr}_R\,\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \qquad \mathrm{perm}_L\,\frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \mathrm{perm}_R\,\frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi}$$

Cut

$$\mathrm{cut}\,\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Example of derivation

$$
\begin{prooftree}
\AxiomC{$B \Rightarrow B$}
\LeftLabel{$\mathrm{wk_L}$}
\UnaryInfC{$A, B \Rightarrow B$}
\AxiomC{$A \Rightarrow A$}
\RightLabel{$\mathrm{wk_L}$}
\UnaryInfC{$B, A \Rightarrow A$}
\RightLabel{$\mathrm{perm_L}$}
\UnaryInfC{$A, B \Rightarrow A$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$A, B \Rightarrow B \wedge A$}
\RightLabel{$\wedge_L^1$}
\UnaryInfC{$A \wedge B, B \Rightarrow B \wedge A$}
\RightLabel{$\mathrm{perm_L}$}
\UnaryInfC{$B, A \wedge B \Rightarrow B \wedge A$}
\RightLabel{$\wedge_L^2$}
\UnaryInfC{$A \wedge B, A \wedge B \Rightarrow B \wedge A$}
\RightLabel{$\mathrm{wk_L}$}
\UnaryInfC{$A \wedge B \Rightarrow B \wedge A$}
\RightLabel{$\rightarrow_R$}
\UnaryInfC{$\Rightarrow A \wedge B \rightarrow B \wedge A$}
\end{prooftree}
$$

Soundness. If $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow A$ then $\Gamma \vdash_{\mathrm{CP}} A$.

Proof sketch. By showing that the rules of **G1cp** preserve theoremhood.

Completeness. If $\Gamma \vdash_{\mathrm{CP}} A$ then $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow A$.

Proof sketch. By deriving the axioms and simulating the rules of the Hilbert system.

$$\text{MP}\ \frac{\vdash_{\mathrm{CP}} A \quad \vdash_{\mathrm{CP}} A \to B}{\vdash_{\mathrm{CP}} B} \quad \rightsquigarrow \quad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$\vee_L \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$$

- $A \vee B$ principal (or main) formula
- A, B active (or secondary) formulas
- Γ, Δ context

Analytic rule All formulas occurring in the premisses are subformulas of formulas occurring in the conclusion (typically, the active formulas are subformulas of the principal formula)

Strictly analytic rule The rule is analytic, moreover the premisses have a lower complexity than the conclusion

Analytic calculus All rules of the calculus are analytic

Remark.

- All logical rules are strictly analytic
- Contraction is not strictly analytic

$$\mathrm{ctr_L}\,\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathrm{ctr_R}\,\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

- Cut is not analytic (the cut formula A can be any formula)

$$\mathrm{cut}\,\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

CUT IS ELIMINABLE: **G1cp** $=$ **G1cp** $\setminus \{\mathrm{cut}\}$

*With abuse of notation, we write **G1cp** for both the calculus and the set of sequents derivable in it

This means that all sequents derivable in **G1cp** are derivable without use of cut

Hard to stress enough the importance of cut elimination

"The safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato."

A.N. Whitehead – *Process and Reality*, 1979

The safest general characterization of the proof theory is that it consists of a series of corollaries to cut elimination

Many equivalent formulations of the sequent calculus for CPL

- ☞ In this course we consider G3-style sequent calculi, where sequents are pairs $\Gamma \Rightarrow \Delta$ of finite multisets of formulas
- ☞ In a multiset, the order of formulas does not matter, whereas their multiplicity matters

Example.

$$A \Rightarrow B, C \quad = \quad A \Rightarrow C, B$$
$$A \Rightarrow B, C \quad \neq \quad A \Rightarrow B, C, C$$

Notation

First, let us fix some notation

- $p, q, r, \ldots$ propositional variables.
- p also used to denote any propositioanl variable
- $\perp$ logical constant for *false*

Formulas: $\quad A, B \quad ::= \quad p \mid \perp \mid \neg A \mid A \wedge B \mid A \vee B \mid A \rightarrow B$

- $A, B, C, D, \ldots$ denote any formula
- $\Gamma, \Delta, \Sigma, \Pi, \ldots$ denote any finite multiset of formulas

Complexity

$$c(\perp) = 0; c(p) = 1; c(\circ A) = c(A) + 1; c(A \circ B) = c(A) + c(B) + 1$$
$$c(A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m) = c(A_1) + \cdots + c(A_n) + c(B_1) + \cdots + c(B_m)$$

Sequent $\Gamma \Rightarrow \Delta$ Γ, Δ finite multisets of formulas

G3cp

Initial sequents

$$\text{init} \quad p, \Gamma \Rightarrow \Delta, p \qquad\qquad \bot_{\mathsf{L}} \quad \bot, \Gamma \Rightarrow \Delta$$

Logical rules

$$\neg_{\mathsf{L}} \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad \neg_{\mathsf{R}} \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$\wedge_{\mathsf{L}} \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{R}} \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \qquad \vee_{\mathsf{L}} \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$$

$$\vee_{\mathsf{R}} \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \rightarrow_{\mathsf{L}} \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_{\mathsf{R}} \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Sequent $\Gamma \Rightarrow \Delta$ $\qquad \Gamma, \Delta$ finite multisets of formulas

G3cp

Initial sequents $\quad$ init $p, \Gamma \Rightarrow \Delta, p$ $\qquad \perp_L \ \perp, \Gamma \Rightarrow \Delta$

Logical rules

$$\neg_L \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad \neg_R \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

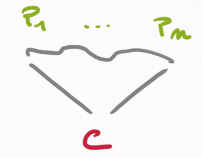
$$\wedge_L \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_R \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \qquad \vee_L \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$$

$$\vee_R \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \rightarrow_L \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

- Logical rules only
- What about the structural rules?

Admissibility, eliminability, invertibility

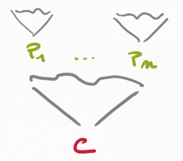
Derivable rule $P_1, \ldots, P_n / C$
There is a derivation tree for the conclusion where every leaf is labelled with an initial sequents or a premiss P_i of the rule



Admissible rule $P_1, \ldots, P_n / C$
If each premiss is derivable, then the conclusion is also derivable

Remark.

- Derivability entails admissibility



- If a rule R is admissible in **G3cp**, then **G3cp** $+ R =$ **G3cp**

☞ Adding an admissible rule to the calculus does not extend the set of derivable sequents

Theorem. All structural rules and cut are admissible in **G3cp**

G1cp

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{wk_L}\;\dfrac{B \Rightarrow B}{A, B \Rightarrow B} \qquad \dfrac{\dfrac{A \Rightarrow A}{B, A \Rightarrow A}\,\mathrm{wk_L}}{A, B \Rightarrow A}\,\mathrm{wk_L}}{A, B \Rightarrow B \wedge A}\,\wedge_{\mathrm{R}}}{A \wedge B, B \Rightarrow B \wedge A}\,\wedge^{1}_{\mathrm{L}}}{B, A \wedge B \Rightarrow B \wedge A}\,\mathrm{perm_L}}{A \wedge B, A \wedge B \Rightarrow B \wedge A}\,\wedge^{2}_{\mathrm{L}}}{A \wedge B \Rightarrow B \wedge A}\,\mathrm{wk_L}}{\Rightarrow A \wedge B \rightarrow B \wedge A}\,\rightarrow_{\mathrm{R}}
$$

G3cp

$$
\dfrac{\dfrac{\dfrac{A, B \Rightarrow B^{*} \qquad A, B \Rightarrow A^{*}}{A, B \Rightarrow B \wedge A}\,\wedge_{\mathrm{R}}}{A \wedge B \Rightarrow B \wedge A}\,\wedge_{\mathrm{L}}}{\Rightarrow A \wedge B \rightarrow B \wedge A}\,\rightarrow_{\mathrm{R}}
$$

The application of structural rules is never needed in **G3cp**
This ensures practical advantages that we will see and use later on

$^{*}A, \Gamma \Rightarrow \Delta, A$ is not an initial sequent but is derivable for every A
(*Exercise.* Prove this claim by induction on the construction of A)

Two possible approaches

R admissible in calculus C Define C without R, show $C + R = C$

R eliminable in calculus C Define C with R, show that $C – R = C$

☞ For our purposes, the two approaches are equivalent

Different ways of proving rule admissibilty/elimination

- Show that the calculus without the rule is complete (e.g. w.r.t. the intended semantics)
- Provide an effective procedure to eliminate every application of the rule in a derivation

Proof of cut admissibilty/eliminability via effective cut elimination procedure are always long and sometimes rather tricky.

A cut elimination procedure transforms a derivation $\mathcal{D}$ into an equivalent derivation $\mathcal{D}'$ that doesn't contain applications of cut

- starting from the topmost applications of cut
- replacing it with one or more "lower" applications of cut: at a lower level, or on a cut formula of lower complexity
- recursively lower all applications of cut
- eliminate applications of cut at the leaves or on atomic formulas
- repeat until all applications of cut have been eliminated

☞ Many cases to consider, depending on whether the cut formula is principal in both, or one, or none of the last rules applied in the derivations of the premisses of cut, considering for each case all rules that have been possibly applied

Example. Consider the following application of cut, with the cut formula $A \wedge B$ principal in the last rule applications of both premisses of cut, with the subderivations $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ cut-free.

$$
\dfrac{
\dfrac{\dfrac{\mathcal{D}}{\Gamma \Rightarrow \Delta, A} \qquad \dfrac{\mathcal{D}'}{\Gamma \Rightarrow \Delta, B}}{\Gamma \Rightarrow \Delta, A \wedge B}\,\wedge_R
\qquad
\dfrac{\dfrac{\mathcal{D}''}{A, B, \Gamma' \Rightarrow \Delta'}}{A \wedge B, \Gamma' \Rightarrow \Delta'}\,\wedge_L
}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}
$$

The derivation is transformed into the following one, with two applications of cut with a cut formula of lower complexity

$$
\dfrac{
\dfrac{
\dfrac{\mathcal{D}'}{\Gamma \Rightarrow \Delta, B}
\qquad
\dfrac{\dfrac{\mathcal{D}}{\Gamma \Rightarrow \Delta, A} \qquad \dfrac{\mathcal{D}''}{A, B, \Gamma' \Rightarrow \Delta'}}{B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}
}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\,\text{cut}
}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{ctr}^*
$$

Remark.

Permutation rules trivialised by the multiset data structure

$$\rightsquigarrow \quad A, B, \Gamma \Rightarrow \Delta \text{ is the same as } B, A, \Gamma \Rightarrow \Delta$$

In CPL, one can do the same with contraction defining Γ, Δ as sets

$$\rightsquigarrow \quad A, A, \Gamma \Rightarrow \Delta \text{ becomes the same as } A, \Gamma \Rightarrow \Delta$$

However, contraction will not be always admissible in sequent calculi for modal logics

- ☞ Set-based sequents are not always adequate to define calculi for modal logics
- ☞ Better to study the admissibility of contraction explicitly

More precisely, weakening and contraction are height-preserving admissible in **G3cp**

Height-preserving admissible (single-premiss) rule (hp-admissible)
If the premiss has a derivation of height h, then the conclusion has a derivation of height $\leq h$

(*Derivation height = length of the longest branch)

Theorem. Left and right weakening and contraction are hp-admissible in **G3cp**

Invertible rule If the conclusion is derivable, then the premisses are also derivable

☞ The conclusion of an invertible rule is derivable if and only if its premisses are derivable

Example. Derivable conclusion and derivable premisses

$$\frac{p, q \Rightarrow p \vee q}{p \wedge q \Rightarrow p \vee q}\,\wedge_{\mathrm{L}} \qquad \frac{p \wedge q \Rightarrow p, q}{p \wedge q \Rightarrow p \vee q}\,\vee_{\mathrm{R}}$$

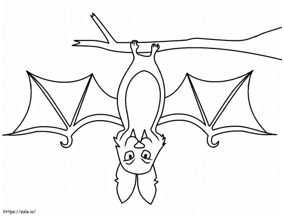
Non-derivable conclusion and non-derivable premisses

$$\frac{p \Rightarrow p \wedge q \qquad q \Rightarrow p \wedge q}{p \vee q \Rightarrow p \wedge q}\,\vee_{\mathrm{L}} \qquad \frac{p \vee q \Rightarrow p \qquad p \vee q \Rightarrow q}{p \vee q \Rightarrow p \wedge q}\,\wedge_{\mathrm{R}}$$

Height-preserving invertible rule (hp-invertible)
If the conclusion is derivable with a derivation of height h, then the premisses are also derivable with derivations of height $\leq h$

Theorem. All rules of **G3cp** are hp-invertible

Backward proof-search



Backwards applicable rule

A rule R is backwards applicable to a sequent $\Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is the conclusion of an instance of R.

Example.

Both $\wedge_{\mathrm{L}}$ and $\vee_{\mathrm{R}}$ are backwards applicable to $A \wedge B \Rightarrow B \vee A$

$$\wedge_{\mathrm{L}} \frac{A, B \Rightarrow B \vee A}{A \wedge B \Rightarrow B \vee A} \uparrow \qquad\qquad \vee_{\mathrm{R}} \frac{A \wedge B \Rightarrow B, A}{A \wedge B \Rightarrow B \vee A} \uparrow$$

Backwards applicable rule

A rule R is backwards applicable to a sequent $\Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is the conclusion of an instance of R

Backward (or root-first, or bottom-up) proof search for $\Gamma \Rightarrow \Delta$

The construction of a derivation tree from the root to the leaves such that the root is labelled with the sequent $\Gamma \Rightarrow \Delta$, and the branches are expanded by applying at each step a backwards applicable rule

Proof of $\Gamma \Rightarrow \Delta$

The tree generated by a backward proof search for $\Gamma \Rightarrow \Delta$

Failed proof

A proof where some leaves are not initial sequents and that cannot be further expanded

Successful proof-search for $\Gamma \Rightarrow \Delta$ constructs a derivation

Example. Failed proof

$$\uparrow \quad \dfrac{\dfrac{\dfrac{\overset{\checkmark}{p, q \Rightarrow q} \qquad \overset{\times}{p, q \Rightarrow r}}{p, q \Rightarrow q \wedge r}\,\wedge_R}{p \wedge q \Rightarrow q \wedge r}\,\wedge_L}{\Rightarrow p \wedge q \to q \wedge r}\,\to_R$$

$p, q \Rightarrow r$ is not an initial sequent, and no rule is backward applicable to it

Example. Successfull proof-search = Derivation

$$\uparrow \quad \dfrac{\dfrac{\dfrac{\overset{\checkmark}{p, q \Rightarrow q} \qquad \overset{\checkmark}{p, q \Rightarrow p}}{p, q \Rightarrow q \wedge p}\,\wedge_R}{p \wedge q \Rightarrow q \wedge p}\,\wedge_L}{\Rightarrow p \wedge q \to q \wedge p}\,\to_R$$

Some properties of **G3cp**

that are our desiderata for modal logic calculi

Formula interpretation

$$i(\Gamma \Rightarrow \Delta) := \bigwedge \Gamma \to \bigvee \Delta$$

with $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bot$

A sequent derivation can be seen as a sequence of validity-preserving trasformations of formulas

$$\dfrac{\dfrac{\dfrac{A, B \Rightarrow B \qquad A, B \Rightarrow A}{A, B \Rightarrow B \wedge A}\wedge_R}{A \wedge B \Rightarrow B \wedge A}\wedge_L}{\Rightarrow A \wedge B \to B \wedge A}\to_R \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{A \wedge B \to B \qquad A \wedge B \to A}{A \wedge B \to B \wedge A}}{A \wedge B \to B \wedge A}}{\top \to (A \wedge B \to B \wedge A)}$$

Syntactic completeness

$$\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \vdash_{\mathrm{CP}} \bigwedge \Gamma \rightarrow \bigvee \Delta.$$

Valid sequent

$$\models \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \begin{array}{l} \text{for every valuation } v, \\ \text{if } v \Vdash A \text{ for all } A \in \Gamma, \\ \text{then } v \Vdash B \text{ for some } B \in \Delta \end{array}$$

Semantic completeness

$$\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \models \Gamma \Rightarrow \Delta.$$

Termination of backward proof-search
Backward proof-search always terminates after a finite number of steps

Termination is ensured by: **Strict analiticity**

- The premisses of every rule have a lower complexity than the conclusion

☞ Complexity of sequent is always reduced by bottom-up rule applications

- The premisses of every rule only contain subformulas of formulas in the conclusion

☞ All formulas occurring in a derivation of $\Gamma \Rightarrow \Delta$ are subformulas of formulas occuring in $\Gamma \Rightarrow \Delta$

Decision procedure by a single proof-search

A single proof-search for $\Gamma \Rightarrow \Delta$ is sufficient to establish whether $\Gamma \Rightarrow \Delta$ is derivable

Ensured by: **Invertibility of the rules**

- For every rule, the conclusion is derivable if and only if the premisses are derivable
- ☞ Backward rule applications preserve (in)derivability of sequents
- ☞ In every proof-search tree, the root is derivable if and only if all leaves are derivable

Remark.
With invertible rules, the order of rule applications is irrelevant

Example.

$$\dfrac{\dfrac{\dfrac{p, q \Rightarrow q \qquad p, q \Rightarrow r}{p, q \Rightarrow q \wedge r}\,\wedge_{\mathrm{R}}}{p \wedge q \Rightarrow q \wedge r}\,\wedge_{\mathrm{L}}}{\Rightarrow p \wedge q \to q \wedge r}\,\to_{\mathrm{R}}$$

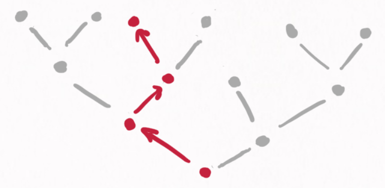
is equivalent to

$$\dfrac{\dfrac{\dfrac{p, q \Rightarrow q}{p \wedge q \Rightarrow q}\,\wedge_{\mathrm{L}} \qquad \dfrac{p, q \Rightarrow r}{p \wedge q \Rightarrow r}\,\wedge_{\mathrm{L}}}{p \wedge q \Rightarrow q \wedge r}\,\wedge_{\mathrm{R}}}{\Rightarrow p \wedge q \to q \wedge r}\,\to_{\mathrm{R}}$$

Bottom-up proof-search in **G3cp** provides an optimal decision procedure for CPL

- ▶ Every bottom-up rule application deletes one connective
- ☞ Length of proof branches linearly bounded by the length of the root sequent

NP decision procedure for underivability (= satisfiability)
Backward proof-search, at each step non-deterministically choose an applicable rule and the correct premiss, until an initial sequent or a failed proof is obtained



Semantic reading of sequents

Formulas on the left as true, formulas on the right as false

Backward semantic reading of sequent rules

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}$$

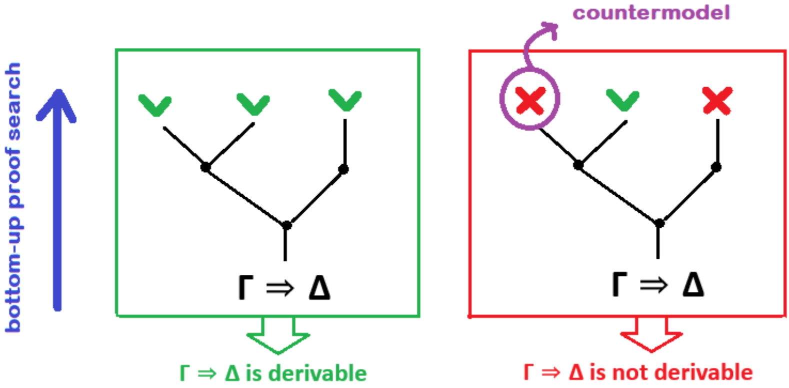
If $A \rightarrow B$ is true, then A is false or B is true

☞ Sequent derivations as uspide down tableaus

A failed derivation constructs a countermodel of the root sequent

$$\dfrac{\dfrac{\dfrac{p, q \Rightarrow q \qquad p, q \Rightarrow r}{p, q \Rightarrow q \wedge r}\wedge_R}{p \wedge q \Rightarrow q \wedge r}\wedge_L}{\Rightarrow p \wedge q \rightarrow q \wedge r}\rightarrow_R \quad \rightsquigarrow \quad \left.\begin{array}{r} q \mapsto 1 \\ p \mapsto 1 \\ r \mapsto 0 \end{array}\right\} \; p \wedge q \rightarrow q \wedge r \mapsto 0$$

☞ (The leaf of) a single failed branch suffices to construct a countermodel



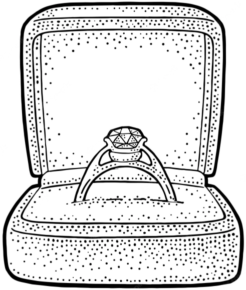
Exercise. Take any propositional formula and check whether it is valid using backward proof-search. If it not, define a countermodel of it on the basis of the failed proof

Bonus property for families of logics

Modularity
The calculi for the stronger systems are defined by adding rules to the calculi for the weaker systems, without modifying the basic rules

- Dosen principle [Došen, 1985], [Wansing, 1994]. Fixed, basic rules for $\Box$, extensions captured by additional structural rules
- Practical advantage. No need to entirely re-prove cut elimination for each single logic, just consider the additional rules

Modal logic



$$A, B ::= p \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \Box A \mid \Diamond A$$

$$A, B ::= p \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

CP	axiomatisation of classical prop. logic
dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
k	$\Box(A \to B) \to (\Box A \to \Box B)$
nec	if A is provable, so is $\Box A$

The S5 cube of normal modal logics

$$A, B ::= p \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

CP	axiomatisation of classical prop. logic
dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
k	$\Box(A \to B) \to (\Box A \to \Box B)$
nec	if A is provable, so is $\Box A$
d	$\Box A \to \Diamond A$
t	$\Box A \to A$
b	$A \to \Box\Diamond A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$

The S5 cube of normal modal logics

$$A, B ::= p \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

CP	axiomatisation of classical prop. logic
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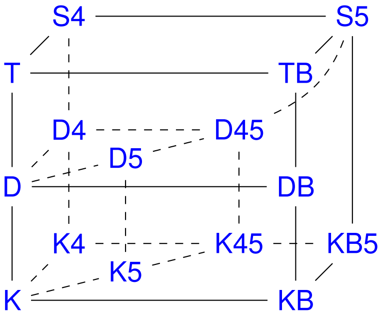
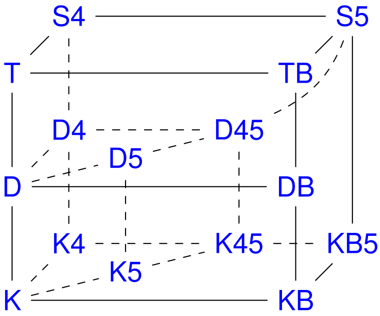
nec	if A is provable, so is $\Box A$
d	$\Box A \to \Diamond A$
t	$\Box A \to A$
b	$A \to \Box\Diamond A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$



$$A, B ::= p \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

CP	axiomatisation of classical prop. logic
dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
k	$\Box(A \to B) \to (\Box A \to \Box B)$
nec	if A is provable, so is $\Box A$
d	$\Box A \to \Diamond A$
t	$\Box A \to A$
b	$A \to \Box\Diamond A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$



☞ $\vdash_X A \rightsquigarrow A$ is derivable from the axioms of X

☞ $\Gamma \vdash_X A \rightsquigarrow \bigwedge \Gamma \to A$ is derivable from the axioms of X

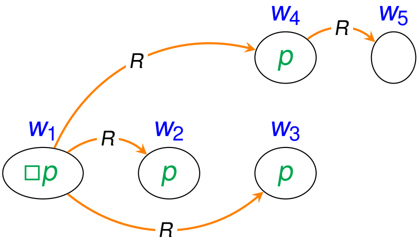
$$\mathcal{M} = \langle W, R, v \rangle$$

- W non-empty set of elements (*worlds*)
- R binary relation on W (*accessibility relation*)
- v valuation function $atm \longrightarrow W$

Satisfiability $\mathcal{M}, w \Vdash A$

$$\mathcal{M}, w \Vdash \Box A \quad \textit{iff} \quad \text{for all } u \text{ s.t. } wRu, u \Vdash A$$

$$\mathcal{M}, w \Vdash \Diamond A \quad \textit{iff} \quad \text{there exists } u \text{ s.t. } wRu \text{ and } u \Vdash A$$



Name	Axiom	Frame condition	
d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Reflexivity	$\forall x (xRx)$
b	$A \to \Box\Diamond A$	Symmetry	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Transitivity	$\forall x \forall y \forall z ((xRy \wedge yRz) \to xRz)$
5	$\Diamond A \to \Box\Diamond A$	Euclideaness	$\forall x \forall y \forall z ((xRy \wedge xRz) \to yRz)$

Notation. We denote $\mathcal{X}$ the class of all models satisfying all conditions corresponding to the axioms of the logic X

Validity in a model

$$\mathcal{M} \models A \quad \textit{iff} \quad \text{for all } w \in \mathcal{M},\ \mathcal{M}, w \Vdash A$$

Validity in a class of models

$$\models_{\mathcal{X}} A \quad \textit{iff} \quad \text{for all } \mathcal{M} \in \mathcal{X},\ \mathcal{M} \models A$$

Logical consequence

$$\Gamma \models_{\mathcal{X}} A \quad \textit{iff} \quad \begin{array}{l} \text{for all } \mathcal{M} \in \mathcal{X}, \text{ for all } w \in \mathcal{M}, \\ \textit{if } \ \mathcal{M}, w \Vdash B \text{ for all } B \in \Gamma, \textit{ then } \ \mathcal{M}, w \Vdash A \end{array}$$

Completeness. $\Gamma \vdash_{\mathsf{X}} A$ iff $\Gamma \models_{\mathcal{X}} A$ [BdRV, 2001]

Satisfiability problem for X

$$\exists \mathcal{M} \in \mathcal{X}, \exists w \in \mathcal{M} : \mathcal{M}, w \Vdash A \ ?$$

Validity problem for X

$$\forall \mathcal{M} \in \mathcal{X}, \mathcal{M} \models A \ ?$$

Derivability problem for X

$$\vdash_X A \ ?$$

☞ A valid in $\mathcal{X}$ iff $\neg A$ not satisfiable in $\mathcal{X}$

☞ A valid in $\mathcal{X}$ iff A derivable in X (if X complete w.r.t. $\mathcal{X}$)

Complexity of satisfiability problem.
The satisfiablity problem for all logics between K and S4 is PSPACE-complete, the satisfiability problem for all logics with axiom 5 is NP-complete [Ladner, 1977], [Halpern, Rêgo, 2007]

Sequent calculi for modal logic

$$\Box \Rightarrow \Diamond$$

Sequent calculi for modal logic extend **G3cp** with rules for the modalities

Example. The sequent calculus for K

$$\textbf{G3cp} \; + \; {}_{\mathrm{k}}\frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \quad \text{with } n \geq 0$$

Sequent calculi for modal logic extend **G3cp** with rules for the modalities

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Exercise. Derive the axiom k and the rule of necessitation

Sequent calculi for modal logic extend **G3cp** with rules for the modalities

Example. The sequent calculus for K

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Exercise. Derive the axiom k and the rule of necessitation

Notation

Given $\Sigma = B_1, \ldots, B_n$, we define $\Box\Sigma = \Box B_1, \ldots, \Box B_n$.

The rule k can be written as

$$\mathrm{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

The rule

$$\mathsf{k}\ \frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

is adequate in the language with $\Box$ only

If both $\Box$ and $\Diamond$ are primitive, one has to replace k with the two rules

$$\mathsf{k}_{\Box}\ \frac{\Sigma \Rightarrow A, \Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Diamond\Pi, \Delta} \qquad \mathsf{k}_{\Diamond}\ \frac{\Sigma, A \Rightarrow \Pi}{\Gamma, \Box\Sigma, \Diamond A \Rightarrow \Diamond\Pi, \Delta}$$

☞ We adopt the more standard approach of defining rules for $\Box$ only.

[Ohnishi, Matsumoto, 1957], [Fitting, 1983], [Takano, 1992]

$$\mathsf{k}\ \frac{\Gamma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \qquad \mathsf{t}\ \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

$$4\ \frac{\Box\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \qquad 45\ \frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

- Sequent calculus for K: **G3cp** $+$ k
- Sequent calculus for T: **G3cp** $+$ k $+$ t
- Sequent calculus for S4: **G3cp** $+ 4 +$ t
- Sequent calculus for S5: **G3cp** $+ 45 +$ t

Exercise. Derive the axioms t, 4, 5 in the respective calculi

We say that **G3X** is a calculus for the logic X if it is complete w.r.t. X

Syntactic completeness

$$\vdash_{\mathbf{G3X}} \ \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \vdash_{X} \ \bigwedge \Gamma \rightarrow \bigvee \Delta$$

Sequent $\Gamma \Rightarrow \Delta$ valid in a model $\mathcal{M}$

$$\mathcal{M} \models \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \begin{array}{l} \forall w \text{ of } \mathcal{M}, \text{ if } \forall A \in \Gamma, \mathcal{M}, w \Vdash A, \\ \text{then } \exists B \in \Delta \text{: } \mathcal{M}, w \Vdash B \end{array}$$

Sequent $\Gamma \Rightarrow \Delta$ valid in $\mathcal{X}$

$$\models_{\mathcal{X}} \ \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \forall \mathcal{M} \in \mathcal{X}, \ \mathcal{M} \models \Gamma \Rightarrow \Delta$$

Semantic completeness

$$\vdash_{\mathbf{G3X}} \ \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \models_{\mathcal{X}} \ \Gamma \Rightarrow \Delta$$

☞ Syntactic and semantic completeness coincide if X is characterised by $\mathcal{X}$

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; \mathsf{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Soundness

- The initial sequents are valid in $\mathcal{K}$
- The propositional rules preserve validity in $\mathcal{K}$
- The rule k preserves validity in $\mathcal{K}$ (*Exercise.*)

  Semantically

  If $\models_{\mathcal{X}} \Sigma \Rightarrow A$, then $\models_{\mathcal{X}} \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta$

  Syntactically

  If $\vdash_{\mathsf{K}} \bigwedge \Sigma \to A$, then $\vdash_{\mathsf{K}} \bigwedge \Gamma \wedge \bigwedge \Box\Sigma \to \Box A \vee \bigvee \Delta$

☞ All sequents derivable in **G3K** are valid in $\mathcal{X}$

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; \mathrm{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Structural properties

- Weakening and contraction are hp-admissible
- All propositional rules are hp-invertible
  (but not the rule k, see below)
- Cut is admissible

Completeness

- All propositional axioms are derivable
- The axiom k is derivable

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B, A \qquad B, A \Rightarrow B}{A \to B, A \Rightarrow B}\,\to_{\mathsf{L}}}{\Box(A \to B), \Box A \Rightarrow \Box B}\,\mathsf{k}}{\Box(A \to B) \Rightarrow \Box A \to \Box B}\,\to_{\mathsf{R}}}{\Rightarrow \Box(A \to B) \to (\Box A \to \Box B)}\,\to_{\mathsf{R}}$$

- The rule nec is derivable $\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\mathsf{k}$
- Modus ponens is simulated by Cut

$$\dfrac{\Rightarrow A \qquad \dfrac{\Rightarrow A \to B}{A \Rightarrow B}\,\to_{\mathsf{R}}\text{-inv}}{\Rightarrow B}\,\text{cut}$$

or

$$\dfrac{\Rightarrow A \to B \qquad \dfrac{\Rightarrow A \qquad A, A \to B \Rightarrow B}{A \to B \Rightarrow B}\,\text{cut}}{\Rightarrow B}\,\text{cut}$$

☞ If A is derivable in X, then $\Rightarrow A$ is derivable in **G3K**

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; \mathrm{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Some remarks about the rule k

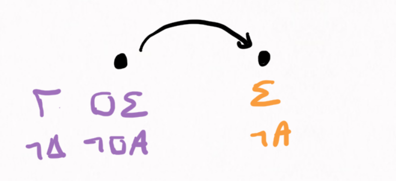
- Context Γ, Δ in the conclusion ensures admissibility of weakening
- No separate left and right rules for $\Box$ (vs. Boolean connectives)
- The rule is parametric, not a single principal formula
- Intuitively, it expresses a global transformation of a derivation of A from assumptions Σ into a derivation of $\Box A$ from assumptions $\Box\Sigma$

A semantic intuition of the rule k

Backward application of k

$$\text{k}\ \frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \uparrow$$

corresponds to moving to an accessible world



Remark. Context is deleted by backward application of k

- ☞ Leads to loss of information
- ☞ This makes the rule k not invertible

Example.

$$\frac{p, q \Rightarrow \bot}{\Box p, \Box q \Rightarrow \Box(p \wedge q), \Box\bot}$$

the conclusion is derivable but the premiss is not

Consequently:

- ☞ One failed proof is not sufficient to ensure non-derivability (might be due to wrong applications of the non invertible rule)
- ☞ Hence, in particular, it does not provide a countermodel
- ☞ Backward proof-search in **G3K** requires backtracking

Possible backward proof-search strategy

- Propositional rules first
- When no propositional rule is applicable, apply k on all $\Box$-formulas on the left
- But, still, one needs to choose one $\Box$-formula on the right

Backward proof-search in **G3K** requires backtracking
But... is this that bad?

- Length of proof banches is polynomially bounded by the size of the root sequent (*exercise*: prove this claim)
- Proof branches can be examined separately. Hence, computation space can be reused

☞ Backward proof-search in **G3K** provides a complexity optimal PSPACE decision prodedure for derivability in K

Remark.
No invertibility, no countermodels, but optimal complexity
☞ As we will see, different calculi can perform different tasks

$$\mathsf{T} \;=\; \mathsf{K} \;+\; \Box A \to A$$

$$\mathbf{G3T} \;=\; \mathbf{G3K} \;+\; \mathrm{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

Remark

- Context is preserved by backward application of t
- t is a “local” rule

$$\textbf{G3T} \;=\; \textbf{G3K} \;+\; \text{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

This calculus is NOT contraction-free complete [Goré, 1999]

Exercise. Prove that

1. The sequent $\Rightarrow \neg\Box(p \wedge \neg\Box p)$ is valid in $\mathcal{T}$
2. The sequent $\Rightarrow \neg\Box(p \wedge \neg\Box p)$ is derivable in **G3T** with contraction
3. The sequent $\Rightarrow \neg\Box(p \wedge \neg\Box p)$ is not derivable in **G3T** without contraction

Possible solution

2. *The sequent* $\Rightarrow \neg\Box(p \wedge \neg\Box p)$ *is derivable in* **G3T** *with contraction*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, \neg\Box p \Rightarrow p}{p \wedge \neg\Box p \Rightarrow p}\wedge_L}{p, \Box(p \wedge \neg\Box p) \Rightarrow \Box p}k}{p, \neg\Box p, \Box(p \wedge \neg\Box p) \Rightarrow}\neg_L}{p \wedge \neg\Box p, \Box(p \wedge \neg\Box p) \Rightarrow}\wedge_L}{\Box(p \wedge \neg\Box p), \Box(p \wedge \neg\Box p) \Rightarrow}t}{\Box(p \wedge \neg\Box p) \Rightarrow}ctr_L}{\Rightarrow \neg\Box(p \wedge \neg\Box p)}\neg_R$$

Possible solution

3. *The sequent* $\Rightarrow \neg\Box(p \wedge \neg\Box p)$ *is not derivable in* **G3T** *without contraction*

Consider the following proof tree obtained from $\Rightarrow \neg\Box(p \wedge \neg\Box p)$ via backward proof-search in **G3T**, where at each step the only applicable rule has been applied, and the leaf is not derivable

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow p}{p \Rightarrow \Box p}\,\mathrm{k}}{p, \neg\Box p \Rightarrow}\,\neg_L}{p \wedge \neg\Box p \Rightarrow}\,\wedge_L}{\Box(p \wedge \neg\Box p) \Rightarrow}\,\mathrm{t}}{\Rightarrow \neg\Box(p \wedge \neg\Box p)}\,\neg_R
$$

Two ways to recover completeness

1. Add explicit contraction

   ☞ Backward proof-search is not terminating

2. Cumulative formulation of the rule t

$$\text{t}\,\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

- ☞ Premiss more complex than the conclusion
- ☞ Backward proof-search is not terminating *per se*
- ☞ Loop-checking (redundancy-checking) required to ensure termination

$$\mathrm{t}\,\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

Condition for backward application of t (loop-checking)
The premiss is not setwise equivalent to a sequent already occurring in the same branch

Example.

$$\mathrm{k}\,\dfrac{\mathrm{t}\,\dfrac{\mathrm{t}\,\dfrac{\Box\Box p, \Box p, p \Rightarrow p \vee q}{\Box\Box p, \Box p \Rightarrow p \vee q}\;\checkmark}{\Box\Box p \Rightarrow p \vee q}}{\Box\Box\Box p, p \Rightarrow \Box(p \vee q)} \qquad \mathrm{k}\,\dfrac{\mathrm{t}\,\dfrac{\mathrm{t}\,\dfrac{\Box\Box p, \Box p, \Box p \Rightarrow p \vee q}{\Box\Box p, \Box p \Rightarrow p \vee q}\;\chi}{\Box\Box p \Rightarrow p \vee q}}{\Box\Box\Box p, p \Rightarrow \Box(p \vee q)}$$

$$\mathrm{S5} = \mathrm{K} + \Box A \to A, \Box A \to \Box\Box A, A \lor \Box\neg\Box A$$

$$\textbf{G3S5} = \textbf{G3cp} + {}_{\mathrm{t}}\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} + {}_{45}\frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

The sequent calculus for S5 is NOT cut-free complete.

Exercise. Prove that

1. The sequent $\Rightarrow p \lor \Box\neg\Box p$ is valid in $\mathcal{S}5$
2. The sequent $\Rightarrow p \lor \Box\neg\Box p$ is derivable in **G3S5** with cut
3. The sequent $\Rightarrow p \lor \Box\neg\Box p$ is not derivable in **G3S5** without cut

Possible solution

2. *The sequent* $\Rightarrow p \vee \Box\neg\Box p$ *is derivable in* **G3S5** *with* cut

$$
\dfrac{\dfrac{\dfrac{\Box p \Rightarrow \Box p}{\Rightarrow \Box p, \neg\Box p}\,\neg_R}{\Rightarrow \Box p, \Box\neg\Box p}\,45 \qquad \dfrac{p \Rightarrow p}{\Box p \Rightarrow p}\,\mathrm{t}}{\dfrac{\Rightarrow p, \Box\neg\Box p}{\Rightarrow p \vee \Box\neg\Box p}\,\vee_R}\,\textcolor{magenta}{\mathrm{cut}}
$$

Possible solution

3. *The sequent* $\Rightarrow p \vee \Box\neg\Box p$ *is not derivable in* **G3S5** *without* cut

Consider the following proof tree obtained from $\Rightarrow p \vee \Box\neg\Box p$ via backward proof-search in **G3S5**, where at each step the only applicable rule has been applied, and the leaf is not derivable

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow}{\Box p \Rightarrow}\,\mathrm{t}}{\Rightarrow \neg\Box p}\,\neg_{\mathrm{R}}}{\Rightarrow p, \Box\neg\Box p}\,45}{\Rightarrow p \vee \Box\neg\Box p}\,\vee_{\mathrm{R}}$$

$$\textbf{G3S5} \quad = \quad \textbf{G3cp} \; + \; {}_{\mathrm{t}}\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \; + \; {}_{45}\frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

- **G3S5** + analytic cut is complete. That is, cut is needed only on subformulas of the end sequent
- No purely Gentzen-style, fully cut-free calculus for S5 is possible [Lellmann, Pattinson, 2013]

Two approaches to extend the basic formalism

1. Enrich the language of the calculus

   ☞ Labelled sequent calculi (Lecture 2)

2. Enrich the structure of sequents

   ☞ Hypersequent calculi, nested calculi (Lectures 3,4)

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