

Proof Theory of Modal Logic

Lecture 2: Labelled Proof Systems

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Partial references:

- ▶ [Kanger, 1957] Spotted formulas for S5
- ▶ [Fitting, 1983], [Goré 1998] Tableaux + labels
- ▶ [Simpson, 1994], [Viganò, 1998] Natural deduction + labels
- ▶ [Mints, 1997], [Viganò, 2000], [Negri, 2005] Sequent calculus + labels

We follow the approach of Negri:

- ▶ *Proof analysis in modal logics* [Negri, 2005]
- ▶ *Contraction-free sequent calculi for geometric theories with an application to Barr's theorem* [Negri, 2003]

- ▶ Labelled sequent calculus for K
- ▶ Frame conditions: a general recipe
- ▶ Semantic completeness *tomorrow*

Labelled sequent calculus for K



$$A, B ::= p \mid \perp \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \Box A \mid \Diamond A$$

Take countably many variables x, y, z, \dots (the **lables**)

Labelled formulas

- ▶ xRy meaning ' x has access to y ' (relational atoms)
- ▶ $x:A$ meaning ' x satisfies A '

Labelled sequent

$$\mathcal{R}, \Gamma \Rightarrow \Delta$$

where

- ▶ \mathcal{R} is a multiset of relational atoms;
- ▶ Γ, Δ are multisets of labelled formulas *without* relational atoms.

Labelled sequents lack a formula interpretation

$$\text{init} \frac{}{\mathcal{R}, x:p, \Gamma \Rightarrow \Delta, x:p}$$

$$\wedge_R \frac{\mathcal{R}, x:A, x:B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:A \wedge B, \Gamma \Rightarrow \Delta}$$

$$\vee_L \frac{\mathcal{R}, x:A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x:B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:A \vee B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_L \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \quad \mathcal{R}, x:B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:A \rightarrow B, \Gamma \Rightarrow \Delta}$$

$$\Box_L \frac{xRy, \mathcal{R}, \overbrace{y:A, x:\Box A}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x:\Box A, \Gamma \Rightarrow \Delta}$$

$$\perp_L \frac{}{\mathcal{R}, x:\perp, \Gamma \Rightarrow \Delta}$$

$$\wedge_L \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x:B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \wedge B}$$

$$\vee_R \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A, x:B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \vee B}$$

$$\rightarrow_R \frac{x:A, \mathcal{R}, \Gamma \Rightarrow \Delta, x:B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \rightarrow B}$$

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, \overbrace{y:A}}{y \text{ fresh}} \frac{}{\mathcal{R}, \Gamma \Rightarrow \Delta, \overbrace{x:\Box A}}$$

$\forall y (xRy \rightarrow y:\#)$

y fresh means y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$ $y \neq x$

$$\begin{array}{c}
 \text{init} \frac{}{\mathcal{R}, x:p, \Gamma \Rightarrow \Delta, x:p} \\
 \wedge_R \frac{\mathcal{R}, x:A, x:B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:A \wedge B, \Gamma \Rightarrow \Delta} \\
 \vee_L \frac{\mathcal{R}, x:A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x:B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:A \vee B, \Gamma \Rightarrow \Delta} \\
 \rightarrow_L \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \quad \mathcal{R}, x:B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:A \rightarrow B, \Gamma \Rightarrow \Delta} \\
 \Box_L \frac{xRy, \mathcal{R}, y:A, x:\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x:\Box A, \Gamma \Rightarrow \Delta} \\
 \Diamond_L \frac{xRy, \mathcal{R}, y:A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x:\Diamond A, \Gamma \Rightarrow \Delta} y \text{ fresh}
 \end{array}
 \qquad
 \begin{array}{c}
 \perp_L \frac{}{\mathcal{R}, x:\perp, \Gamma \Rightarrow \Delta} \\
 \wedge_L \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x:B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \wedge B} \\
 \vee_R \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A, x:B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \vee B} \\
 \rightarrow_R \frac{x:A, \mathcal{R}, \Gamma \Rightarrow \Delta, x:B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \rightarrow B} \\
 \Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y:A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:\Box A} y \text{ fresh} \\
 \Diamond_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x:\Diamond A, y:A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x:\Diamond A}
 \end{array}$$

y fresh means y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

We write $\vdash_{\text{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta$ if there is a derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta$ in labK.

Example: $\vdash_{\text{labK}} \Rightarrow x:(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$

$$\begin{array}{c}
 \begin{array}{c}
 \text{init} \frac{}{xRy, y:p \Rightarrow y:q, x:\Diamond p, y:p} \\
 \hline
 \Diamond_R \frac{}{xRy, y:A \Rightarrow y:q, x:\Diamond p}
 \end{array}
 \quad
 \begin{array}{c}
 \text{init} \frac{}{xRy, x:\Box q, y:q, y:p \Rightarrow y:q} \\
 \hline
 \Box_L \frac{}{xRy, x:\Box q, y:p \Rightarrow y:q}
 \end{array}
 \\
 \hline
 \rightarrow_L \frac{}{xRy, x:\Diamond p \rightarrow \Box q, y:p \Rightarrow y:q}
 \\
 \rightarrow_R \frac{}{xRy, x:\Diamond p \rightarrow \Box q \Rightarrow y:p \rightarrow q}
 \\
 \Box_R \frac{}{x:\Diamond p \rightarrow \Box q \Rightarrow x:\Box(p \rightarrow q)}
 \\
 \rightarrow_R \frac{}{\Rightarrow \underline{x}:(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)}
 \end{array}$$

Γ set of formulas, A formula

HILBERT-STYLE
AXIOM SYSTEM

$$\Gamma \vdash_K A$$



LOGICAL
CONSEQUENCE

$$\Gamma \models A$$



$$\left[\begin{array}{c} \vdash \frac{x: \Gamma \Rightarrow x: A}{\text{labK}} \\ \text{LABELLED S.C.} \end{array} \right]$$

$$x: \Gamma = \{ x: G \mid G \in \Gamma \}$$

Given a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$, and a model $\mathcal{M} = \langle W, R, v \rangle$, let $\text{Lb}(\mathcal{S}) = \{x \mid x \in \mathcal{R} \cup \Gamma \cup \Delta\}$, and $\rho : \text{Lb}(\mathcal{S}) \rightarrow W$ (interpretation).

Satisfiability of labelled formulas at \mathcal{M} under ρ :

$$\mathcal{M}, \rho \models xRy \quad \text{iff} \quad \rho(x)R\rho(y)$$

$$\mathcal{M}, \rho \models x:A \quad \text{iff} \quad \mathcal{M}, \rho(x) \models A$$

Satisfiability of sequents at \mathcal{M} under ρ (φ is xRy or $x:A$):

$$\mathcal{M}, \rho \models \mathcal{R}, \Gamma \Rightarrow \Delta \quad \text{iff}$$

if for all $\varphi \in \mathcal{R} \cup \Gamma$ it holds that $\mathcal{M}, \rho \models \varphi$,

then for some $x:D \in \Delta$ it holds that $\mathcal{M}, \rho \models x:D$.

A sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ has a countermodel iff there are \mathcal{M}, ρ such that:

- ▷ $\mathcal{M}, \rho \models \varphi$, for all $\varphi \in \mathcal{R} \cup \Gamma$, and
- ▷ $\mathcal{M}, \rho \not\models x:D$, for all $x:D \in \Delta$.

Validity of sequents in a class of frames \mathcal{X} :

$$\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \text{iff} \quad \text{for any } \rho \text{ and any } \mathcal{M} \in \mathcal{X}, \mathcal{M}, \rho \models \mathcal{R}, \Gamma \Rightarrow \Delta$$

Theorem (Soundness). If $\vdash_{\text{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models \mathcal{R}, \Gamma \Rightarrow \Delta$

Proof. By induction on the height of the derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta$.

We need to show that initial sequents are valid (easy) and that inference rules preserve validity.

Case of \Box_R :

To prove: If $\mathcal{R}, \Gamma \Rightarrow \Delta$, $x: \Box A$ is not valid, then $xRy, \mathcal{R}, \Gamma \Rightarrow \Delta$, $y: A$ is not valid.

Assume $\mathcal{R}, p \not\models \mathcal{R}, \Gamma \Rightarrow \Delta$, $x: \Box A$, for some $\mathcal{R} = \langle w, R, v \rangle$ and p .

$\rightarrow \triangleright \mathcal{R}, p \models \varphi$ for all $\varphi \in \mathcal{R} \cup \Gamma$

$\rightarrow \triangleright \mathcal{R}, p \not\models z: D$, for all $z: D \in \Delta$

$\triangleright \mathcal{R}, p \not\models x: \Box A \leadsto \mathcal{R}, p(x) \not\models \Box A \leadsto$ there is w s.t. $p(x) R w$ and $\mathcal{R}, w \not\models A$.

Take $p'(y) = w$, and $p'(k) = p(k)$, for all $k \neq y$ in $\mathcal{R} \cup \Gamma \cup \Delta$.

It is easy to verify that

$\mathcal{R}, p' \not\models xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y: A$

$\mathcal{R}, p' \not\models y: A$

□

Γ set of formulas, A formula

HILBERT-STYLE
AXIOM SYSTEM

$\Gamma \vdash_K A$



$\vdash x : \Gamma \Rightarrow x : A$

$\text{labK} \cup \{\text{cut}\}$
LABELLED S.C.



LOGICAL
CONSEQUENCE

$\Gamma \models A$



SOUNDNESS

$\vdash x : \Gamma \Rightarrow x : A$

labK
LABELLED S.C.

cut - admissibility



Substitution on labelled formulas:

$$xRy[z/y] := xRz$$

$$y:A[z/y] := z:A$$

Substitution on multisets of labelled formulas $\Gamma[z/y]$

Lemma (Substitution). Rule subst is hp-admissible.

$$\text{subst} \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}[y/x], \Gamma[y/x] \Rightarrow \Delta[y/x]}$$

Lemma (Weakening). Rules wk_L , wk_R are hp-admissible (φ is xRy or $x:A$).

$$\text{wk}_L \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\varphi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{wk}_R \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi}$$

Lemma (Invertibility).

For every rule r , if the conclusion of r is derivable with a derivation of height h , then each of its premisses is derivable, with at most the same h .

Proof (sketch):

$$\frac{xRy, R, \Gamma \Rightarrow \Delta, y:A}{R, \Gamma \Rightarrow \Delta, x:\Box A} \Box_R$$

If $\vdash_R R, \Gamma \Rightarrow \Delta, x:\Box A$, then for every label y not occurring in $R \cup \Gamma \cup \Delta$, we have that $\vdash_R xRy, R, \Gamma \Rightarrow \Delta, y:A$.

$$\frac{xRy, R, y:A, \Box A, \Gamma \Rightarrow \Delta}{xRy, R, x:\Box A, \Gamma \Rightarrow \Delta} \Box_L$$

If $\vdash_R xRy, R, x:\Box A, \Gamma \Rightarrow \Delta$ then $\vdash_R xRy, R, y:A, x:\Box A, \Gamma \Rightarrow \Delta$.

Lemma (Contraction). Rules ctr_L , ctr_R are hp-admissible (φ is xRy or $x:A$).

$$\text{ctr}_L \frac{\varphi, \varphi, R, \Gamma \Rightarrow \Delta}{\varphi, R, \Gamma \Rightarrow \Delta}$$

$$\text{ctr}_R \frac{R, \Gamma \Rightarrow \Delta, \varphi, \varphi}{R, \Gamma \Rightarrow \Delta, \varphi}$$

Lemma (Cut). The cut rule is admissible.

$$\text{cut} \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \quad x:A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $(c(A), h_1 + h_2)$.

$$\begin{array}{c} \frac{\frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y:A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x:\Box A} \Box_R \quad \frac{xRz, \mathcal{R}', x:\Box A, z:A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}', x:\Box A, \Gamma' \Rightarrow \Delta'} \Box_L}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{cut} \end{array}$$

$$\begin{array}{c} \frac{\frac{xRz, \mathcal{R}, \Gamma \Rightarrow \Delta, z:A}{\mathcal{R}, \mathcal{R}, xRz, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'} \text{cut} \quad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:\Box A \quad xRz, \mathcal{R}', x:\Box A, z:A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}, \mathcal{R}', z:A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{cut}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{ctr}_L, \text{ctr}_R \end{array}$$

For Γ set of formulas and $x:\Gamma = \{x:G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_K A$ then $\vdash_{\text{labK}} x:\Gamma \Rightarrow x:A$.

Γ set of formulas, A formula

HILBERT-STYLE
AXIOM SYSTEM

$$\Gamma \vdash_K A$$

SYNTACTIC
COMPLETENESS



$$\vdash x : \Gamma \Rightarrow x : A$$

labK \cup {cut}
LABELLED S.C.



LOGICAL
CONSEQUENCE

$$\Gamma \models A$$

SOUNDNESS



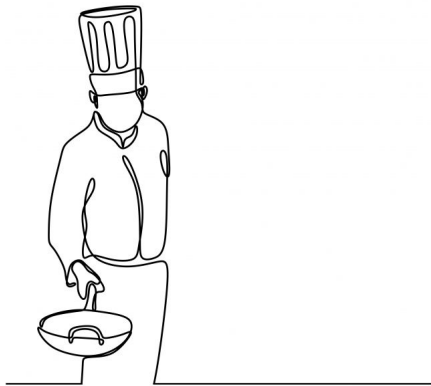
$$\vdash x : \Gamma \Rightarrow x : A$$

labK
LABELLED S.C.



CUT-ADMISSIBILITY

Frame conditions: a general recipe



What do we mean by modularity?

Let $K = \text{CPL} \cup \{k, \text{nec}\}$. Logic K is characterised by the class of all Kripke frames.

Name	Axiom	Frame condition
d	$\Box A \rightarrow \Diamond A$	Seriality $\forall x \exists y (xRy)$
t	$\Box A \rightarrow A$	Reflexivity $\forall x (xRx)$
b	$A \rightarrow \Box \Diamond A$	Symmetry $\forall x \forall y (xRy \rightarrow yRx)$
4	$\Box A \rightarrow \Box \Box A$	Transitivity $\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow xRz)$
5	$\Diamond A \rightarrow \Box \Diamond A$	Euclideaness $\forall x \forall y \forall z ((xRy \wedge xRz) \rightarrow yRz)$

Take $X \subseteq \{d, t, b, 4, 5\}$.

We write $\Gamma \vdash_{K \cup X} A$ iff A is derivable from Γ in the axiom system $K \cup X$.

We denote by \mathcal{X} the class of frames satisfying properties in X .

We write $\Gamma \models_{\mathcal{X}} A$ iff A is logical consequence of Γ in the class of frames \mathcal{X} .

Theorem. For $X \subseteq \{d, t, b, 4, 5\}$, $\Gamma \vdash_{K \cup X} A$ iff $\Gamma \models_{\mathcal{X}} A$.

Name	Axiom	Frame condition	
d	$\Box A \rightarrow \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \rightarrow A$	Reflexivity	$\forall x (xRx)$
b	$A \rightarrow \Box \Diamond A$	Symmetry	$\forall x \forall y (xRy \rightarrow yRx)$
4	$\Box A \rightarrow \Box \Box A$	Transitivity	$\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow xRz)$
5	$\Diamond A \rightarrow \Box \Diamond A$	Euclideaness	$\forall x \forall y \forall z ((xRy \wedge xRz) \rightarrow yRz)$

Frame conditions can be characterised by first-order logic formulas, in the language consisting of a single predicate symbol, $R(x, y)$.

Proof systems for geometric theories, [Negri, 2003]:

“axioms-as-rules”

How to transform axioms of geometric theories (geometric implications) into rules, preserving the structural properties of the calculus.

The first-order logic formulas corresponding to the frame conditions above (and many more!) are geometric implications

A first-order signature is a tuple $\sigma = \langle c, d, \dots, f, g, \dots, p, q, \dots \rangle$

- ▶ Constant symbols c, d, \dots
- ▶ Function symbols f, g, \dots , each with arity > 0
- ▶ Predicate symbols p, q, \dots , each with arity ≥ 0

A first-order language over a signature σ , denoted $\mathcal{L}(\sigma)$, consists of:

- ▶ The **terms** generated from a countably many variables x, y, \dots using the constants and function symbols of σ ;
- ▶ The **formulas** generated from the terms of $\mathcal{L}(\sigma)$ and predicate symbols of σ using the operators $\perp, \wedge, \vee, \rightarrow, \forall, \exists$.

A first-order language with equality over a signature σ , denoted $\mathcal{L}^=(\sigma)$, additionally comprises a binary predicate for equality.

Example.

$\mathcal{L}^=(0, \text{succ}^1, +^2, \times^2)$ is the language of arithmetic

$\mathcal{L}(R^2)$ is the language we use to express frame conditions

Fix a first-order language $\mathcal{L}(\sigma)$ (with or without equality).

A **first-order theory** over $\mathcal{L}(\sigma)$ is a set of closed formulas of $\mathcal{L}(\sigma)$.

Example. Peano Arithmetic and Robinson Arithmetic are first-order theories over $\mathcal{L}^=(0, suc, +, \times)$.

A **geometric formula** is a formula of $\mathcal{L}(\sigma)$ which does not contain \rightarrow or \forall .

A **geometric implication** is closed formula of $\mathcal{L}(\sigma)$ of the shape:

$$\forall \vec{x}(A \rightarrow B), \quad \text{for } A, B \text{ geometric formulas}$$

A **geometric theory** over $\mathcal{L}(\sigma)$ is a first-order theory over $\mathcal{L}(\sigma)$ whose formulas are geometric implications.

Example. Robinson arithmetic is a geometric theory over the language $\mathcal{L}^=(0, suc, +, \times)$.

Geometric implications can be expressed as conjunctions of **geometric axioms**, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall \vec{x} \left(P \rightarrow \left(\exists \vec{y}_1 (Q_1) \vee \dots \vee \exists \vec{y}_m (Q_m) \right) \right)$$

- ▶ $\vec{x}, \vec{y}_1, \dots, \vec{y}_m$ are (possibly empty) vectors of variables; *disjoint*
- ▶ $m \geq 0$;
- ▶ P, Q_1, \dots, Q_m are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
- ▶ $\vec{y}_1, \dots, \vec{y}_m$ do not occur in P .

Geometric implications can be expressed as conjunctions of **geometric axioms**, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall \vec{x} \left(P \rightarrow \left(\exists \underline{\vec{y}}_1 (Q_1) \vee \dots \vee \exists \underline{\vec{y}}_m (Q_m) \right) \right)$$

- ▶ $\vec{x}, \vec{y}_1, \dots, \vec{y}_m$ are (possibly empty) vectors of variables;
- ▶ $m \geq 0$;
- ▶ P, Q_1, \dots, Q_m are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
- ▶ $\vec{y}_1, \dots, \vec{y}_m$ do not occur in P .

Geometric axioms can be turned into sequent calculus rules:

$$\text{GA} \frac{\overline{\Xi}_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{\Xi}_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}$$

$P \rightarrow$

- ▶ Π is the multiset of atomic formulas in P ;
- ▶ $\overline{\Xi}_i$ is the multiset of atomic formulas in Q_i , for each $i \leq m$;
- ▶ $\vec{z}_1, \dots, \vec{z}_m$ do not occur in $\Gamma \cup \Delta$.

Geometric implications can be expressed as conjunctions of **geometric axioms**, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall \vec{x} \left(P \rightarrow \left(\exists \vec{y}_1 (Q_1) \vee \dots \vee \exists \vec{y}_m (Q_m) \right) \right)$$

- ▶ $\vec{x}, \vec{y}_1, \dots, \vec{y}_m$ are (possibly empty) vectors of variables;
- ▶ $m \geq 0$;
- ▶ P, Q_1, \dots, Q_m are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
- ▶ $\vec{y}_1, \dots, \vec{y}_m$ do not occur in P .

Geometric axioms can be turned into sequent calculus rules:

$$\text{GA} \frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \dots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}$$

$\tau_{R(x,y)}$

n_{Ry}

- ▶ Π is the multiset of atomic formulas in P ;
- ▶ Ξ_i is the multiset of atomic formulas in Q_i , for each $i \leq m$;
- ▶ $\vec{z}_1, \dots, \vec{z}_m$ do not occur in $\Gamma \cup \Delta$.

$$\forall \vec{x} \left(P \rightarrow \left(\exists \vec{y}_1 (Q_1) \vee \dots \vee \exists \vec{y}_m (Q_m) \right) \right)$$

$$\text{GA} \frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \dots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\begin{array}{c} \text{P} \qquad \qquad \text{Q} \\ \hline \forall x y z (x R y \wedge y R z \rightarrow x R z) \end{array} \quad \frac{x R z, x R y, y R z, R, \Gamma \Rightarrow \Delta}{x R y, y R z, R, \Gamma \Rightarrow \Delta} \text{tr}$$

$$\begin{array}{c}
 \text{ser} \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \quad y \text{ fresh} \quad \text{ref} \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{sym} \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \\
 \\
 \text{tr} \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{euc} \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}
 \end{array}$$

$$\begin{array}{c}
 \text{ser} \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \quad y \text{ fresh} \quad \text{ref} \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{sym} \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \\
 \\
 \text{tr} \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{euc} \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}
 \end{array}$$

For $X \subseteq \{d, t, b, 4, 5\}$, $\text{labK} \cup X$ is defined by adding to labK the rules for frame conditions corresponding to elements of X , plus the rules obtained by to satisfy the **closure condition** (contracted instances of the rules):

$$\text{euc} \frac{yRy, xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \text{euc}' \frac{yRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Example: $\text{labK} \cup \{5\}$ denotes the proof system $\text{labK} \cup \{\text{euc}, \text{euc}'\}$.

We denote by $\vdash_{\text{labK} \cup X} S$ derivability of labelled sequent S in $\text{labK} \cup X$.

For $X \subseteq \{d, t, b, 4, 5\}$:

Theorem (Soundness). If $\vdash_{\text{labK} \cup X} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models_X \mathcal{R}, \Gamma \Rightarrow \Delta$.

Example. If the premiss of rule *ser* is valid in all serial models, then its conclusion is valid in all serial models.

$$\text{ser} \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} y \text{ fresh}$$

Lemma (Cut). The cut rule is admissible in $\text{labK} \cup X$:

$$\text{cut} \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x:A \quad x:A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

For Γ set of formulas and $x:\Gamma = \{x:G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_{K \cup X} A$ then $\vdash_{\text{labK} \cup X} x:\Gamma \Rightarrow x:A$.

- **Systems of rules** [Negri, 2016], to capture theories / logics characterized by generalized geometric implications:

$$GA_0 = \forall \vec{x} \left(P \rightarrow \left(\exists \vec{y}_1 (Q_1) \vee \cdots \vee \exists \vec{y}_m (Q_m) \right) \right)$$

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[Negri, 2005]: labelled proof system for GL!

labK \cup X: main results

$$X \subseteq \{d, t, b, u, s\}$$

Γ set of formulas, A formula

HILBERT-STYLE
AXIOM SYSTEM

$$\Gamma \vdash_{K \cup X} A$$

SYNTACTIC
COMPLETENESS

$$\vdash x : \Gamma \Rightarrow x : A$$

labK \cup X \cup {cut}
LABELLED S.C.



LOGICAL
CONSEQUENCE

$$\Gamma \models_X A$$

SOUNDNESS



CUT-ADMISSIBILITY

$$\vdash x : \Gamma \Rightarrow x : A$$

labK \cup X
LABELLED S.C.

- ▶ Derive axiom 4, that is, $\Box A \rightarrow \Box \Box A$, in $\text{labK} \cup \{t, 5\}$. Then, show that rule tr is derivable in $\text{labK} \cup \{t, 5\} \cup \{wk_L, wk_R\}$.
- ▶ Derive axiom 5, that is, $\Diamond A \rightarrow \Box \Diamond A$, in $\text{labK} \cup \{b, 4\}$. Then, show that rule euc is derivable in $\text{labK} \cup \{b, 4\} \cup \{wk_L, wk_R\}$.
- ▶ Write down the labelled rule corresponding to the frame condition of confluence:

$$\forall x, y, z \left((R(x, y) \wedge R(x, z)) \rightarrow \exists k (R(y, k) \wedge R(z, k)) \right)$$

- ▶ Write down the sequent calculus rules corresponding to the axioms of Robinson Arithmetic. Can we use the results from [Negri, 2003] to prove consistency of Robinson Arithmetic? If yes, how?