Proof Theory of Modal Logic

Lecture 3, part 1: Labelled Proof Systems

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The plan

- Labelled sequent calculus for K
- Frame conditions: a general recipe
- Semantic completeness

Recap

- Frame conditions as geometric axioms (FO-formulas in $\mathcal{L}(R)$)

- "Axioms-as-rules" method [Negri, 2003]
  Geometric axioms can be turned into sequent calculus rules
  (general result to define cut-free sequent calculi
  for geometric theories)

- We can define cut-free labelled sequent calculi for modal logics whose frame conditions can be expressed as geometric axioms [Negri, 2005].

From geometric axioms to rules [Negri, 2003]

Geometric implications can be expressed as conjunctions of geometric axioms, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall\vec{x}\Big(P \to \big(\exists\vec{y}_1(Q_1) \vee \cdots \vee \exists\vec{y}_m(Q_m)\big)\Big)$$

- $\vec{x}, \vec{y}_1, \ldots, \vec{y}_m$ are (possibly empty) vectors of disjoint variables;
- $m \geq 0$;
- $P, Q_1, \ldots, Q_m$ are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
- $\vec{y}_1, \ldots, \vec{y}_m$ do not occur in P.

Geometric axioms can be turned into sequent calculus rules:

$$\frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}\ \text{GA}$$

- Π is the multiset of atomic formulas in P;
- Ξ_i is the multiset of atomic formulas in Q_i, for each $i \leq m$;
- $\vec{z}_1, \ldots, \vec{z}_m$ do not occur in $\Gamma \cup \Delta$.

From geometric axioms to labelled rules [Negri, 2003]

Geometric implications can be expressed as conjunctions of geometric axioms, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

Take $\mathcal{L}(R)$ as our language

$R(x, y)$

$$\forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \lor \cdots \lor \exists \vec{y}_m(Q_m)\big)\Big)$$

- $\vec{x}, \vec{y}_1, \ldots, \vec{y}_m$ are (possibly empty) vectors of disjoint variables;
- $m \geq 0$;
- P, $Q_1, \ldots, Q_m$ are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
- $\vec{y}_1, \ldots, \vec{y}_m$ do not occur in P.

Geometric axioms can be turned into sequent calculus rules:

$$\frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}\ \text{GA}$$

this becomes a STRUCTURAL RULE: a rule on relational atoms

- Π is the multiset of atomic formulas in P;
- Ξ_i is the multiset of atomic formulas in Q_i, for each $i \leq m$;
- $\vec{z}_1, \ldots, \vec{z}_m$ do not occur in $\Gamma \cup \Delta$.

Examples

$\bigwedge \emptyset := \top$

$\bigvee \emptyset := \bot$

$$\forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big)$$

$$\text{GA}\ \frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

▷ reflexivity

$\forall x (xRx) \rightsquigarrow \forall x (\emptyset \to xRx)$

$$\frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ ref$$

▷ euclidean

$\forall x y z\ (xRy \wedge xRz \to yRz)$

$$\frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ euc$$

▷ seriality

$\forall x \exists y (xRy) \rightsquigarrow \forall x (\emptyset \to \exists y (xRy))$

$$\frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ ser,\ y \text{ fresh}$$

▷ density

$\forall x y\ (xRy \to \exists z (xRz \wedge zRy))$

$$\frac{xRz, zRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}\ den \quad z \text{ fresh}$$

Labelled calculi for the S5-cube [Negri, 2005]

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \text{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

For $X \subseteq \{d, t, b, 4, 5\}$, $\mathrm{labK} \cup X$ is defined by adding to labK the rules for frame conditions corresponding to elements of X, plus the rules obtained by to satisfy the closure condition (contracted instances of the rules):

$z := y$

$$\text{euc}\ \frac{yRy, xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \text{euc}'\ \frac{yRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Example: $\mathrm{labK} \cup \{5\}$ denotes the proof system $\mathrm{labK} \cup \{\mathrm{euc}, \mathrm{euc}'\}$.

We denote by $\vdash_{\mathrm{labK} \cup X} \mathcal{S}$ derivability of labelled sequent $\mathcal{S}$ in $\mathrm{labK} \cup X$.

$$\frac{\{\Xi_i[z_i/y_i], P_1, .., P, P, \dots P_m, \mathcal{R}, \Gamma \Rightarrow \Delta\}_{i \leq m}}{P_1, .., P, P, \dots P_m, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \frac{\{\Xi_i[z_i/y_i], P_1, .., P, \dots P_m, \mathcal{R}, \Gamma \Rightarrow \Delta\}_{i \leq m}}{P_1, .., P, \dots P_m, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Theorem (Soundness). If $\vdash_{\mathsf{labK}\cup\mathsf{X}} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta$.

Example. If the premiss of rule ser is valid in all serial models, then its conclusion is valid in all serial models.

$$\mathsf{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

Lemma (Cut). The cut rule is admissible in labK ∪ X:

$$\mathsf{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \qquad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

For Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_{\mathsf{K}\cup\mathsf{X}} A$ then $\vdash_{\mathsf{labK}\cup\mathsf{X}} x{:}\Gamma \Rightarrow x{:}A$.

- Systems of rules [Negri, 2016], to capture theories / logics characterized by generalized geometric implications:

$$\begin{aligned} GA_0 &= \forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big) \\ GA_1 &= \forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(\bigwedge GA_0) \vee \cdots \vee \exists \vec{y}_m(\bigwedge GA_0)\big)\Big) \\ GA_{n+1} &= \forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(\bigwedge GA_{k_1}) \vee \cdots \vee \exists \vec{y}_m(\bigwedge GA_{k_m})\big)\Big) \end{aligned}$$

  for $k_1, \ldots, k_m \geq n$

  Systems of rules cover all systems of normal modal logics axiomatised by Sahlqvist formulas.

- Gödel-Löb provability logic (GL):

  Transitivity: R is transitive

  Converse well-foundedness: there are no infinite R-chains

  [Negri, 2005]: labelled proof system for GL!

labK ∪ X: main results

$X \subseteq \{d, t, b, 4, 5\}$

Γ set of formulas, A formula

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$\begin{tikzcd}
\Gamma \vdash_{K \cup X} A \arrow[rr, Leftrightarrow] \arrow[d, "\text{SYNTACTIC COMPLETENESS}"'] & & \Gamma \cancel{\models_X} A \arrow[d, dashed, shift right, "\text{completeness}"'] \\
\vdash x : \Gamma \Rightarrow x : A \arrow[rr, "\text{CUT-ADMISSIBILITY}"'] & & \cancel{\vdash} x : \Gamma \Rightarrow x : A \arrow[u, shift right, "\text{SOUNDNESS}"']
\end{tikzcd}$$

labK ∪ X ∪ {cut}
LABELLED S.C.

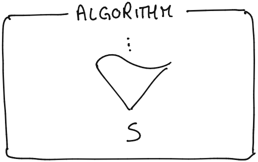
labK ∪ X
LABELLED S.C.

Semantic completeness

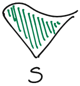
For $X \subseteq \{d, t, b, 4, 5\}$:

Theorem (Proof or Countermodel). For $\mathcal{S}$ labelled sequent, either $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$ or $\mathcal{S}$ has a countermodel satisfying the frame conditions in X.

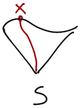
Proof (sketch). Define a proof search algorithm (algorithm implementing proof search in labK ∪ X):



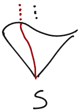
the algorithm produces a labK ∪ X proof



the algorithm produces a finite failed branch, or does not terminate



finite counterm. for S



infinite counterm. for S

For $X \subseteq \{d, t, b, 4, 5\}$,
Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Semantic completeness). If $\Gamma \models_{\mathcal{X}} A$ then $\vdash_{\text{labK}\cup\text{X}} x{:}\Gamma \Rightarrow x{:}A$.

Proof. We prove the contrapositive:

If $\nvdash_{\text{labK}\cup\text{X}} \underline{x{:}\Gamma \Rightarrow x{:}A}$, then $\Gamma \not\models_X A$.

By the Proof or Countermodel Theorem, $x{:}\Gamma \Rightarrow x{:}A$ has a countermodel: there are $\mathcal{M}^X \in X$ and ρ^X such that:

- $\mathcal{M}^X, \rho^X \models x{:}G$, for all $x{:}G \in \Gamma$, and
- $\mathcal{M}^X, \rho^X \not\models x{:}A$.

By definition:

- $\mathcal{M}^X, \rho^X(x) \models G$, for all $G \in \Gamma$, and
- $\mathcal{M}^X, \rho^X(x) \not\models A$.

By definition of logical consequence, $\Gamma \not\models_X A$.

$\square$

Proof search algorithm for labK ∪ X [Negri, 2009]

0. Given a sequent $\mathcal{S}_0$, place $\mathcal{S}_0$ at the root of $\mathcal{T}$.
1. For every rule R $\in \{\wedge_L, \wedge_R, \vee_L, \vee_R, \rightarrow_L, \rightarrow_R, \Box_L, \Box_R, \Diamond_L, \Diamond_R\}$, apply the following:
   a) If every topmost sequent of $\mathcal{T}$ is initial, terminate.
      $\rightsquigarrow$ $\mathcal{S}_0$ is provable in labK ∪ X, and $\mathcal{T}$ defines a labK ∪ X proof for it.
   b) Otherwise, write above each non-initial sequent $\mathcal{S}_i$ of $\mathcal{T}$ the sequent(s) obtained by exhaustively apply rule R to $\mathcal{S}_i$.
2. For every rule R $\in \{\mathrm{ref}, \mathrm{tr}, \mathrm{sym}, \mathrm{ser}, \mathrm{euc}\}$ in labK ∪ X (if any), apply the following:
   a) If every topmost sequent of $\mathcal{T}$ is initial, terminate.
      $\rightsquigarrow$ $\mathcal{S}_0$ is provable in labK ∪ X, and $\mathcal{T}$ defines a labK ∪ X proof for it.
   b) Otherwise, write above each non-initial sequent $\mathcal{S}_i$ of $\mathcal{T}$ the sequent(s) obtained by exhaustively apply rule R to $\mathcal{S}_i$.
3. If there is a topmost sequent $\mathcal{S}_i$ of $\mathcal{T}$ which is non-initial and to which none of the steps in 1 and 2 applied, then terminate.
   $\rightsquigarrow$ $\mathcal{S}_0$ is not provable in labK ∪ X, and the branch $\mathcal{B}^\times$ of $\mathcal{T}$ to which $\mathcal{S}_i$ belongs defines a countermodel for $\mathcal{S}_0$.
   Otherwise, go to step 1.

Theorem (Proof or Countermodel). For $\mathcal{S}$ labelled sequent, either $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$ or $\mathcal{S}$ has a countermodel satisfying the frame conditions in $\mathcal{X}$.

Proof. Run the proof search algorithm for labK $\cup$ X taking $\mathcal{S}_0 = \mathcal{S}$. Then:

- If the algorithm terminates in Step 1 or Step 2, then $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$.
- If the algorithm terminates in Step 3: We construct a countermodel for $\mathcal{S}$ from the finite branch $\mathscr{B}^\times$ produced by the algorithm.
- If the algorithm does not terminate, then all branches of $\mathcal{T}$ are infinite. We construct a countermodel for $\mathcal{S}$ from any infinite branch $\mathscr{B}^\times$ of $\mathcal{T}$.

Let $\mathcal{B}^{\times} = (\mathcal{R}_i, \Gamma_i \Rightarrow \Delta_i)_{i<k}$ be a finite branch in $\mathcal{T}$ produced by the algorithm ($k \in \mathbb{N}$), or an infinite branch in $\mathcal{T}$ ($k = \omega$).
In both cases, $\mathcal{S} = \mathcal{R}_0, \Gamma_0 \Rightarrow \Delta_0$.

We construct a countermodel $\mathcal{M}^{\times}$ from $\mathcal{B}^{\times}$ as follows:

- $W^{\times} = \{x \mid x \text{ occurs in } \mathcal{B}^{\times}\}$;
- $xR^{\times}y$ iff xRy occurs in $(\mathcal{R}_i)_{i<k}$;
- $v^{\times}(p) = \{x \mid x{:}p \text{ occurs in } (\Gamma_i)_{i<k}\}$.

It is easy to verify that $\mathcal{M}^{\times}$ satisfies the frame conditions $\mathcal{X}$.

Truth Lemma. Take $\rho^{\times}(x) = x$, for each label x occurring in $\mathcal{B}^{\times}$. Then:

- If $x{:}A \in (\Gamma_i)_{i<k}$, then $\mathcal{M}^{\times}, \rho^{\times} \models x{:}A$
- If $x{:}A \in (\Delta_i)_{i<k}$, then $\mathcal{M}^{\times}, \rho^{\times} \not\models x{:}A$

Therefore, $\mathcal{M}^{\times}, \rho^{\times} \not\models \mathcal{S}$.

Example

Proof search for $\Rightarrow 0{:}\Diamond\Box p$ in $\mathsf{labK} \cup \{\mathsf{t}, 4\}$

$$\vdots$$

$$\dfrac{0R2, 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p}{2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}\ \Diamond_R$$

$$\dfrac{2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}{2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}\ \mathsf{tr}$$

$$\dfrac{2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}{1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}\ \mathsf{ref}$$

$$\dfrac{1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}{1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p}\ \Box_R$$

$$\dfrac{1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p}{1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}\ \Diamond_R$$

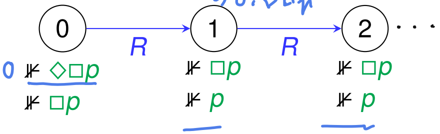
$$\dfrac{1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}{0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}\ \mathsf{ref}$$

$$\dfrac{0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}{0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p}\ \Box_R$$

$$\dfrac{0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p}{0R0 \Rightarrow 0{:}\Diamond\Box p}\ \Diamond_R$$

ref

$\Rightarrow 0{:}\Diamond\Box p$



Bounding proof search

Sources of non-termination, 1

$$\Box_L\,\dfrac{\Box_L\,\dfrac{\Box_L\,\dfrac{\Box_L\,\dfrac{\vdots}{1{:}2,\ 1{:}\Box q,\ 2{:}q,\ 2{:}q,\ 2{:}q \Rightarrow}}{1{:}2,\ 1{:}\Box q,\ 2{:}q,\ 2{:}q \Rightarrow}}{1{:}2,\ 1{:}\Box q,\ 2{:}q \Rightarrow}}{1{:}2,\ 1{:}\Box q \Rightarrow}$$

1R2

$$\text{ser}\,\dfrac{\text{ser}\,\dfrac{\text{ser}\,\dfrac{\text{ser}\,\dfrac{\vdots}{2R3,\ 1R2,\ 0R1 \Rightarrow 0{:}p}}{1R2,\ 0R1 \Rightarrow 0{:}p}}{0R1 \Rightarrow 0{:}p}}{\Rightarrow 0{:}p}$$

Sources of non-termination, 2

$$
\begin{array}{lc}
 & \vdots \\
\hline
\Box_R & 0R2, 1R2, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p \\
\hline
\Diamond_R & 0R2, 1R2, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\
\hline
\mathrm{tr} & 1R2, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\
\hline
\Box_R & 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p \\
\hline
\Diamond_R & 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p \\
\hline
\Box_R & 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p \\
\hline
\Diamond_R & 0R0 \Rightarrow 0{:}\Diamond\Box p \\
\hline
\mathrm{ref} & \Rightarrow 0{:}\Diamond\Box p
\end{array}
$$

In the literature:

- [Negri, 2005]: Minimality argument for some logics in the S5-cube (K, T, S4, S5);
- [Negri, 2014]: Termination for intermediate logics;
- [Garg, Genovese and Negri, 2012]: Termination for multi-modal logics (without symmetry).

As a case study, we shall consider labK $\cup \{t, 4\}$, shortened in labS4.

Theorem (Proof or Finite Countermodel). For $\mathcal{S} = x{:}\Gamma \Rightarrow x{:}A$ labelled sequent, either $\vdash_{\mathrm{labS4}} \mathcal{S}$ or $\mathcal{S}$ has a finite countermodel satisfying ref, tr.

A cumulative version of labS4: labS4^c

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_R\ \frac{\mathcal{R}, x{:}A \wedge B, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad \vee_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, \boxed{x{:}A \vee B}, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, \boxed{x{:}A \vee B}}$$

$$\wedge_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

$$\vee_L\ \frac{\mathcal{R}, x{:}A \vee B, x{:}A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x{:}A \vee B, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \vee B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_L\ \frac{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}A \rightarrow B, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_R\ \frac{\mathcal{R}, x{:}A, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B}$$

$$\Box_L\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, \underline{x{:}\Box A}, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, \underline{x{:}\Box A}}\ y \text{ fresh}$$

$$\Diamond_L\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Diamond A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \Diamond_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A, y{:}A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A}$$

- Rules should be applied exhaustively
- Rules shouldn't be applied redundantly
- We need to limit applications of $\Box_R, \Diamond_L$

Intuitively: A rule application R is redundant at a sequent $\mathcal{S}$ if $\mathcal{S}$ already contains the formulas that would be introduced in one premiss of R.

Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

(ref) If x occurs in $\mathcal{S}$, then xRx occurs in $\mathcal{R}$;

(tr) If xRy and yRz occur in $\mathcal{R}$, then xRz occurs in $\mathcal{R}$;

($\wedge_L$) If $x{:}A \wedge B$ occurs in Γ, then both $x{:}A$ and $x{:}B$ occur in Γ;

($\wedge_R$) If $x{:}A \wedge B$ occurs in Δ, then $x{:}A$ occurs in Δ or $x{:}B$ occur in Δ;

($\Box_L$) If xRy occurs in $\mathcal{R}$ and $x{:}\Box A$ occurs in Γ, then $y{:}A$ occurs in Γ;

($\Box_R$) If $x{:}\Box A$ occurs in Δ, then there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

- Rules should be applied exhaustively
- Rules shouldn't be applied redundantly
- We need to limit applications of $\Box_R, \Diamond_L$

$k \sim x$ iff
- $\{G \mid k:G \in \Gamma\} = \{G \mid x:G \in \Gamma\}$
- $\{D \mid k:D \in \Delta\} = \{D \mid x:D \in \Delta\}$

Intuitively: A rule application R is redundant at a sequent $\mathcal{S}$ if $\mathcal{S}$ already contains the formulas that would be introduced in one premiss of R.

Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

(ref) If x occurs in $\mathcal{S}$, then xRx occurs in $\mathcal{R}$;

(tr) If xRy and yRz occur in $\mathcal{R}$, then xRz occurs in $\mathcal{R}$;

($\wedge_L$) If $x{:}A \wedge B$ occurs in Γ, then both $x{:}A$ and $x{:}B$ occur in Γ;

($\wedge_R$) If $x{:}A \wedge B$ occurs in Δ, then $x{:}A$ occurs in Δ or $x{:}B$ occur in Δ;

($\Box_L$) If xRy occurs in $\mathcal{R}$ and $x{:}\Box A$ occurs in Γ, then $y{:}A$ occurs in Γ;

($\Box_R$) If $x{:}\Box A$ occurs in Δ, then either

a) there is a k such that kRx occurs in $\mathcal{R}$ and $k \sim x$; otherwise

b) there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

If a) holds, we say that x is a $\Box$-copy of k at $\mathcal{S}$.

Does $\Gamma \models_{\{\mathsf{ref},\mathsf{tr}\}} A$ hold?

0. Place $\mathcal{S}_0 = x{:}\Gamma \Rightarrow x{:}A$ at the root of $\mathcal{T}$.
1. For every topmost sequent $\mathcal{S}_i$ of $\mathcal{T}$, apply as much as possible non-redundant instances of the rules: $\mathsf{ref}, \mathsf{tr}, \wedge_{\mathsf{L}}, \wedge_{\mathsf{R}}, \vee_{\mathsf{L}}, \vee_{\mathsf{R}}, \rightarrow_{\mathsf{L}}, \rightarrow_{\mathsf{R}}, \Box_{\mathsf{L}}, \Diamond_{\mathsf{R}}$.
2. If every topmost sequent of $\mathcal{T}$ is initial, terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is provable in labS4.
3. Otherwise, pick a non-initial topmost sequent $\mathcal{S}_k$ of $\mathcal{T}$.
   a) If there are non-redundant $\Box_{\mathsf{R}}$- or $\Diamond_{\mathsf{L}}$- rule instances that can be applied, apply one such instance. Go to Step 1.
   b) Otherwise terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is not provable in labS4.

A countermodel $\mathcal{M}^\times$ for a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ which is non-initial and to which only redundant rules can be applied is defined as follows:

- $W^\times = \{x \mid x \text{ occurs in } \mathcal{S}\}$;
- To define $R^\times$, first define:
  - $xR^\times_1 y$ iff xRy occurs in $\mathcal{R}$;
  - $xR^\times_2 k$ $\cancel{kR^\times_2 x}$ iff x is a $\Box$-copy (or $\Diamond$-copy) of k.

  $\mathcal{R}^\times$ is the reflexive and transitive closure of $R^\times_1 \cup R^\times_2$.
- $v^\times(p) = \{x \mid x{:}p \text{ occurs in } \Gamma\}$.

It is easy to verify that $\mathcal{M}^\times$ satisfies the frame conditions ref, tr.

Truth Lemma. Take $\rho^\times(x) = x$, for each label x occurring in $\mathcal{S}$. Then:

- If $x{:}A$ occurs in Γ, then $\mathcal{M}^\times, \rho^\times \models x{:}A$
- If $x{:}A$ occurs in Δ, then $\mathcal{M}^\times, \rho^\times \not\models x{:}A$

Example   *This example is not correct: see next 2 pages*

Does $\models_{\{ref,tr\}} \Diamond\Box p$ hold?

$$0R2, 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p$$

$\Diamond_R$

$$2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p$$

tr

$$2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p$$

ref

$$1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p$$

$\Box_R$

$$1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p$$

$\Diamond_R$

$$1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p$$

ref

$$0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p$$

$\Box_R$

$$0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p$$

$\Diamond_R$

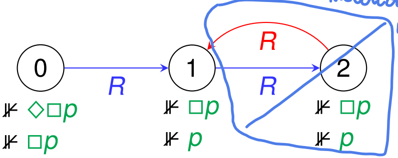
$$0R0 \Rightarrow 0{:}\Diamond\Box p$$

ref

$$\Rightarrow 0{:}\Diamond\Box p$$

Proof search would stop here: the application of $\Box_R$ to $1{:}\Box p$ is redundant, because of b): $1R1$ and $1{:}p \in \Delta$

not needed!



Example # 1

Previous example, correct version: rule $\Box_R$ to $1{:}\Box p$ is redundant because of b); proof search stops.

Does $\models_{\{\text{ref,tr}\}} \Diamond\Box p$ hold?

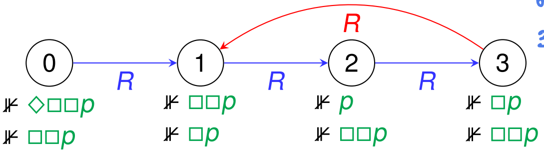
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p}{1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}\,\Diamond_R}{0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}\,\text{ref}}{0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p}\,\Box_R}{0R0 \Rightarrow 0{:}\Diamond\Box p}\,\Diamond_R}{\Rightarrow 0{:}\Diamond\Box p}\,\text{ref}
$$

$$
\begin{array}{ccc}
0 & \xrightarrow{\quad R \quad} & 1 \\
\not\Vdash \Diamond\Box p & & \not\Vdash \Box p \\
\not\Vdash \Box p & & \not\Vdash p
\end{array}
$$

Example # 2: Does $\models_{\{\mathrm{ref},\mathrm{tr}\}} \Diamond\Box\Box p$ hold?

$$
\begin{array}{rc}
 & 0R3, 3R3, 2R3, 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p, 2{:}\Box\Box p, 3{:}\Box p, 3{:}\Box\Box p \\ \hline
\Diamond_R & 0R3, 3R3, 2R3, 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p, 2{:}\Box\Box p, 3{:}\Box p \\ \hline
\mathrm{tr} & 3R3, 2R3, 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p, 2{:}\Box\Box p, 3{:}\Box p \\ \hline
\mathrm{ref} & 2R3, 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p, 2{:}\Box\Box p, 3{:}\Box p \\ \hline
\Box_R & 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p, 2{:}\Box\Box p \\ \hline
\Diamond_R & 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p \\ \hline
\mathrm{tr} & 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p \\ \hline
\mathrm{ref} & 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p, 2{:}p \\ \hline
\Box_R & 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p, 1{:}\Box\Box p \\ \hline
\Diamond_R & 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p \\ \hline
\mathrm{ref} & 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p, 1{:}\Box p \\ \hline
\Box_R & 0R0 \Rightarrow 0{:}\Diamond\Box\Box p, 0{:}\Box\Box p \\ \hline
\Diamond_R & 0R0 \Rightarrow 0{:}\Diamond\Box\Box p \\ \hline
\mathrm{ref} & \Rightarrow 0{:}\Diamond\Box\Box p
\end{array}
$$

rule $\Box_R$ to $3{:}\Box\Box p$ is redundant because of a); 3 is a $\Box$-copy of 1.



Termination. The algorithm terminates in a finite number of steps, yielding either a proof or a sequent from which a countermodel can be extracted.

Theorem (Proof or Finite Countermodel). For $\mathcal{S} = x{:}\Gamma \Rightarrow x{:}A$ labelled sequent, either $\vdash_{\mathrm{labS4}} \mathcal{S}$ or $\mathcal{S}$ has a finite countermodel satisfying ref, tr.

Theorem (Semantic completeness). If $\Gamma \models_{\{\mathrm{ref},\mathrm{tr}\}} A$ then $\vdash_{\mathrm{labS4}} x{:}\Gamma \Rightarrow x{:}A$.

Corollary. S4 has the finite model property.

Corollary. The validity problem of S4 is decidable.

1k 1S4

Properties of labK ∪ X

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
labK ∪ X	no	yes	yes	yes, for most	yes, easy!	yes

1. Check whether $\models_{\{\text{ref},\text{tr}\}} \Diamond\Box(p \vee \Box(p \rightarrow \bot))$ using the terminating algorithm for S4. If the formula is not valid, produce a countermodel.

2. Let $\mathcal{M}^{\times}$ be the countermodel for a sequent $\mathcal{S}$ as defined in Slide 20. Verify that $\mathcal{M}^{\times}$ satisfies the frame conditions ref, tr.
   Then, for $\rho^{\times}(x) = x$, for each label x occurring in $\mathcal{S}$, verify that the Truth Lemma holds, for the cases:
   - If $x{:}\Box A$ occurs in Γ, then $\mathcal{M}^{\times}, \rho^{\times} \models x{:}\Box A$
   - If $x{:}\Box A$ occurs in Δ, then $\mathcal{M}^{\times}, \rho^{\times} \not\models x{:}\Box A$